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Dynamics on Graphs

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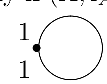
Abstract

The dynamics of a topological Markov chains $\mathcal{D}$ is considered that is defined on a Markov graph, which arises as the oriented line graph from some finite connected graph A with an additional feature: a map $\mathrm{i}_A : \mathrm{E}A \to \mathbb{N}$ from the edges of A to the natural numbers specifies whether a backtracking into an edge $e \in \mathrm{E}A$ is allowed ($\mathrm{i}(e) > 1$) or forbidden ($\mathrm{i}_A(e) = 1$).

Based on covering theory of graphs, geometrical methods are applied to describe the dynamics of $\mathcal{D}$. A tree $\mathcal{T}$, the universal cover of (A, i_A), is constructed along with a group G, the fundamental group, acting by isometries on $\mathcal{T}$ with quotient $A = G\backslash\mathcal{T}$ and projection $\pi : \mathcal{T} \to A$. Under the action of the larger group $\{h \in \mathrm{Is}(\mathcal{T}) : \pi \circ h = \pi\}$, the space of bi-infinite reduced paths $\mathcal{RT}$ is in correspondence with the phase space of $\mathcal{D}$. Using geometry on the covering tree $\mathcal{T}$, a border $\mathcal{T}(\infty)$ is constructed that allows to identify each path of $\mathcal{RT}$ with a pair of distinct border points and an integer number. As an application of these coordinates, α-dimensional densities on $\mathcal{T}(\infty)$ are found for G and will be used to write invariant Markov measures for $\mathcal{D}$, provided that the graph (A, i_A) is unimodular.

These measures enjoy a time-reversal symmetry. Together with the time shift, they are ergodic if and only if $(A, \mathrm{i}_A) \neq (\mathrm{circ}_N, 1)$ for all $N \in \mathbb{N}$. They are mixing if and only if $(A, \mathrm{i}_A) \neq$ 

and (A, i_A) has two closed geodesics of coprime lengths. In case of a minimal indexing (backtracking is allowed into an edge e only if there is no other choice to continue and then $\mathrm{i}(e) = 2$) they resemble the Parry measure, which has an interpretation as the asymptotic distribution of periodic orbits.

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AUTHOR'S DECLARATION

I declare that the work in this dissertation was carried out in accordance with the Regulations of the University of Bristol. The work is original except where indicated by special reference in the text and no part of the dissertation has been submitted for any other degree.
Any views expressed in the dissertation are those of the author and in no way represent those of the University of Bristol.
The dissertation has not been presented to any other University for examination either in the United Kingdom or overseas.

Erlangen, 6 April 2004

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Introduction

> En todas las ficciones, cada vez que un hombre se enfrenta con diversas alternativas, opta por una y elimina las otras; en la del casi inextricable Ts'ui Pên, opta —simultáneamente— por todas. *Crea*, así, diversos porvenires, diversos tiempos, que también proliferan y se bifurcan.
>
> Jorge Luis Borges, El jardín de senderos que se bifurcan (1941)

A graph is a finite selection of real line segments glued together at some of their ends. A particle moves along a segment with constant velocity 1. Once the end of the segment is reached, the particle is scattered with certain probabilities into one of the segments that meet at the same end. In the chosen segment the particle continues moving with constant velocity 1, and so forth. This is how a classical dynamics on a graph is modeled by T. Kottos and U. Smilansky [1]. (Indeed, the main emphasis in their work lies on a quantum mechanical description corresponding to this dynamics.) This probabilistic model shall be replaced by a deterministic dynamics.

The motion of a single particle from the infinitely far away past to the infinitely far away future is described by its trajectory on the graph. A trajectory is written as a map g from the real line $\mathbb{R}$ to the graph. The particle's location at time t is $g(t)$. An observer that moves with this particle at time s will be located at

$$g(s + \Delta t)$$

after a time period of Δt. The change in aspect, from the probabilistic motion to a deterministic motion, is done by considering the space of all possible trajectories of a particle, instead of following one single particle along the time. This space of all possible trajectories is called phase space in the theory of dynamical systems. In the context of Riemannian manifolds, the space of trajectories of a motion along geodesics with constant velocity 1 would be described by the unit tangent bundle of the manifold. This thesis is motivated by the question if there is a description of the phase space of the dynamics on a graph that has a structure similar to the unit tangent bundle of a Riemannian manifold.

At first it shall be remembered what the dynamics looks like in phase space. From a physical perspective, in order to describe a dynamics in terms of trajectories, an observer must be aware of a trajectory in the form of information about the location of its particle at all times, that is in the form of the function g. If he asks a trajectory g about the location of its particle at time s, he will get the information "the particle is at $g(s)$". If he asks the same trajectory about the location at time $s + \Delta t$, the trajectory will give the information "the particle is at $g(s + \Delta t)$". A translation along the real line is an isometry, so there is a trajectory $L_{(\Delta t)}g$ in the phase space defined through

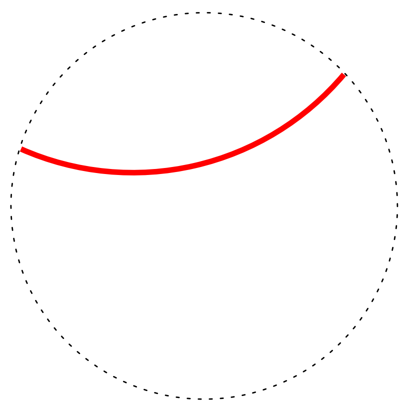
$$\left(L_{(\Delta t)}g\right)(t) := g(\Delta t + t)$$

for all times t. The observer, asking $L_{(\Delta t)}g$ about the location of its particle at time s, will get back the same information as when asking g about the location at time $\Delta t + s$ and both agree for all times s. This is a universal property of all trajectories g. The original dynamics can thus be translated to the phase space: after experiencing the influence of the dynamics for a time period Δt, a phase space element g will be transformed into $L_{\Delta t}(g)$. This is the starting point for an investigation of the system in the context of dynamical systems.

There has been research on this kind of dynamical system by M. Coornaert and A. Papadopoulos in [2] for instance. They considered a graph as a pair, a metric tree together with a group acting by isometries on the tree. For investigation of the geodesic flow they used "Hopf type arguments" that depend on geometry, in this case the geometry on the tree. Among other results they proved ergodicity for certain natural invariant measures.

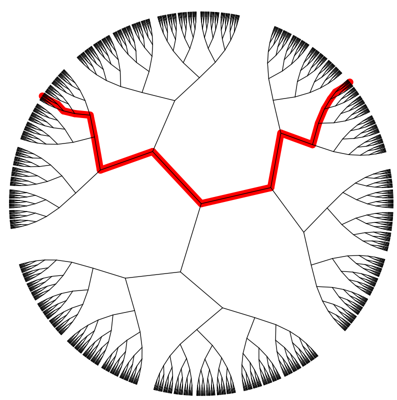
The deterministic point of view presented above is not far from the dynamics we are going to discuss. Using segments of equal length 1, the analysis of the dynamical system simplifies considerably. It is then decomposed by the time-1 map into (topological) factors [3]. In each of these factors the particles are synchronized so as to be located at a constant distance to the previously traversed end. A trajectory is now encoded by a $\mathbb{Z}$-sequence of segment labels together with the information on the direction along which the segment is moved. The dynamics is given by a left shift on these sequences. In the formal graph context that will be used, a segment label with a direction is written as an edge e. The inverse edge $\overline{e}$ refers to the same segment traversed in opposite direction. The invariant measures (Markov measures) that can be used for the phase space of sequences are in a natural correspondence with the probabilities of the probabilistic model which was the starting point of our considerations.

Figure 1: A geodesic in the Poincaré disc.



An important new construction that will be done is motivated by the study of compact Riemannian manifolds of constant negative curvature. These can be described as a quotient of the Poincaré disc $\{z \in \mathbb{C} : |z| < 1\}$ with a group of Möbius transformations acting isometrically on the disk. In a topological sense, the Poincaré disc is the universal cover of the compact manifold, the group is its fundamental group. The free motion of a particle on the compact manifold exhibits strong chaotic features [4]. The corresponding free motion on the disc however is easily described by hyperbolic geometry. Geodesics in the disc are straight lines through the origin or segments of circles perpendicular to the boundary $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Figure 1 shows an example of a geodesic. Indeed a geodesic can be identified with a pair of distinct border points and a real number to provide coordinates for further arguments. See also [5] for an introduction to groups and geometry on the Poincaré disc.

Figure 2: A bi-infinite reduced path in a 3-regular tree.



The analogue to the previous construction can be done for finite connected graphs. Foundations can be found in works [6] by J.-P. Serre and [7] by H. Bass. These find application in the work [8] by M. Burger and S. Mozes for instance. A tree $\mathcal{T}$, the universal cover, will be constructed for a finite connected graph A, along with a group G, the fundamental group. The fundamental group acts by isometries on $\mathcal{T}$ with quotient graph $A = G\backslash\mathcal{T}$ and projection $\pi : \mathcal{T} \to A$. The space $\mathcal{RT}$ of bi-infinite reduced paths on the tree $\mathcal{T}$ is in correspondence with the phase space of the dynamics on the graph A. The action of the so called "full group" $G_f = \{g \in \mathrm{Is}\,(\mathcal{T}) : \pi \circ g = \pi\}$ on $\mathcal{RT}$ provides a bijection. Geometry on the tree $\mathcal{T}$ is used to construct the so called border $\mathcal{T}(\infty)$ for $\mathcal{T}$. Each bi-infinite reduced path can be identified with a pair of distinct border points of the tree and an integer number. Figure 2 displays an example of a bi-infinite reduced path in a three-regular tree.

All that has been indicated in the last paragraph will be presented in detail. Moreover, following ideas from M. Burger and S. Mozes, invariant measures for the dynamical system are derived for the case of a unimodular group G_f. The construction of these measures only involves the tree $\mathcal{T}$ and the group G_f. In most cases, exactly one such measure is produced. This measure will be introduced and explored in the last two sections.

As for further applications of geometrical methods on trees in the mathematical literature, a current interest of research are eigenvalue equations

$$BF = \lambda F$$

for functions F from the vertices of a graph to the complex numbers and for linear operators B on this vector space. A famous operator is the combinatorial Laplacian given by

$$BF(x) = \sum_{\mathrm{d}(x,y)=1} F(y).$$

Here d is the graph theoretic distance (length of the shortest path connecting two vertices), so that B takes sums over adjacent vertices. Issues of research in this area can be found in [9] by A.B. Venkov and A.M. Nikitin or in [10] by A. Terras and D. Wallace. Up to now, both the graph under consideration and a given covering tree for that graph are assumed to be regular with the same degree, i.e. the number of edges with origin at a vertex is independent of the chosen vertex. Important tools for their works are covering theory for graphs, the border of a tree, horocycles, concepts that will be introduced and used here as well.

This Thesis is organised as follows. **Chapter 1** summarizes definitions and basic properties of graphs. Paths will be written by sequences of edges

$$e_1, \ldots, e_n$$

such that $\mathrm{t}(e_i) = \mathrm{o}(e_{i+1})$ for all $1 \leq i < n$. Quotients of graphs by groups are introduced and will be used in a first application to give a positive answer to the question, whether all distance preserving endomorphisms (isometries) of a (locally finite) tree are symmetry maps (automorphisms). The only assumption is the regularity condition of existence of a group of automorphisms producing a finite quotient graph. The statement holds true in a generalization of trees to combinatorial graphs.

Chapter 2 establishes edge-indexed graphs (a function $\mathrm{i}_A : \mathrm{E}A \to \mathbb{N}$ is added to a graph A) together with their distinguished paths, the geodesics. A geodesic is a path, where, whenever a backtracking

$$a, \overline{a}$$

appears in the edge sequence, then

$$\mathrm{i}_A(\overline{a}) > 1.$$

A transition is written from edge-indexed graphs to their oriented line graphs including a correspondence of geodesics with positive paths. This correspondence is used in Chapter 5 for a description of the dynamics as a topological Markov chain. Unimodularity is treated briefly with some examples. By a result of H. Bass and R. Kulkarni in [11], unimodularity of a graph (A, i_A) is equivalent to unimodularity of the group G_f, which will be of importance when writing Markov measures in Section 5.6. The remaining and main contents of the chapter are about connectivity properties. A classification can be used in Section 5.7 to give necessary and sufficient conditions for ergodic properties. The classification is done with elementary arguments.

Chapter 3 introduces covering theory for graphs [6, 7] (there is a more detailed overview at the beginning of the chapter). Fixing a base point x_0 of $\mathrm{V}A$, a universal cover $\mathcal{T}$ can be constructed for a finite connected edge-indexed graph (A, i_A). The tree $\mathcal{T}$ is locally finite and is a multiple cover for A. The fundamental group G acts by isometries on $\mathcal{T}$ with quotient morphism $\pi : \mathcal{T} \to A$ and quotient graph

$$G \backslash \mathcal{T} = A.$$

Locally, the tree $\mathcal{T}$ covers each edge e of A by a number of $\mathrm{i}_A(e)$ edges (all with origin at some vertex $x \in \pi^{-1}(\mathrm{o}(e))$). The full group G_f is defined as $\{h \in \mathrm{Is}\,(\mathcal{T}) : \pi \circ h = \pi\}$ and will play an important role when considering quotients of path spaces in Section 5.2,

$$G_f \backslash \mathcal{R}(\mathcal{T}) = \mathcal{G}(A, \mathrm{i}_A).$$

Topologies on the isometry group of the cover $\mathcal{T}$ and the subgroup G_f are introduced in generalization of ideas by A. Figà-Talamanca and C. Nebbia [12] about regular trees. The group G_f is a locally compact Hausdorff group. Vertex stabilizers are open and compact.

Geometrical methods based on the universal covering tree are founded in **Chapter 4**. Two equivalent descriptions of the border $\mathcal{T}(\infty)$ of a locally finite tree $\mathcal{T}$ are given at the beginning. One of them defines border points as reduced rays having a common initial vertex. The border can be seen as a metric space. It is a complete metric space with "visual" metrics relative to vertices. The different metrics are mutually equivalent. Their topology gives the border the structure of a compact and totally disconnected topological space. The geometrical concept of horocycles for border points ω and a horocycle distance $B_\omega(x, y)$ for vertices will be introduced. They serve in Section 5.3 to assign coordinates to bi-infinite paths (without backtracking) of $\mathcal{T}$ by bijections

$$\kappa_x : \mathcal{R}(\mathcal{T}) \longrightarrow (\mathcal{T}(\infty) \times \mathcal{T}(\infty)) \setminus \mathrm{diag} \times \mathbb{Z}$$

with $\mathrm{diag} = \{(\eta, \eta) \in \mathcal{T}(\infty) \times \mathcal{T}(\infty) : \eta \in \mathcal{T}(\infty)\}$ and vertices x of $\mathcal{T}$. Horocycles will also appear in the definition of α-dimensional densities. The space of their "measurable functions" on the border, the function space of locally constant functions will be introduced at the end of the chapter. However, the densities are written as positive functionals.

The dynamics on a finite connected graph (A, i_A) is formally introduced in Part III, **Chapter 5**. Its correspondence with a topological Markov chain is written. Using the universal cover constructed in Chapter 3, the dynamics will be lifted to the space of reduced bi-infinite paths on the tree. Taking that as a base for further considerations, geometrical arguments are applied to assign triples of coordinates to bi-infinite paths. Such a triple (α, ω, n) consists of two distinct border points α, ω and an integer value n. Positive α-dimensional densities, that are used in the work [8] by M.Burger and S.Mozes in a more general context to write invariant measures directly, are proved in detail to correspond to eigenvectors of a Perron Frobenius matrix. These vectors will be used to write invariant Markov measures for the dynamical system. Examples are provided and some properties of the measures are recorded. They have a time-reversal symmetry. In the case of a minimal edge indexing, the Parry measure is obtained. The last section uses results from Chapter 2 for classifying the considered dynamical systems according to their ergodic properties, whether they are ergodic or mixing.

Part I

Graphs

Chapter 1

A graph tool box

For a better suggestive understanding, an informal description of the graph model may be convenient at the beginning. A graph can be seen as a street map with each carriage way consisting of exactly two lanes, one in each direction. This enables one to speak of the oncoming lane, the beginning and the end of a lane (not of a carriage way).

Turning towards Mathematics, one may model this as a set of edges E (lanes) together with a map $e \mapsto \overline{e}$ (allocation of the oncoming lane) satisfying $\overline{\overline{e}} = e$ and $\overline{e} \neq e$. A pair $\{e, \overline{e}\}$ is called a geometrical edge (carriage way). Origin (beginning) and terminus (end) of edges are called vertices, denoted as a set V. There are maps $\mathrm{o}, \mathrm{t} : \mathrm{E} \to \mathrm{V}$ (assignment of beginning and end) with $\mathrm{o}(e) = \mathrm{t}(\overline{e})$ and $\mathrm{t}(e) = \mathrm{o}(\overline{e})$.

As for further terms we make a choice out of two sources. The graph model used is taken from Serre [6]. The term "reduced path" is taken from Bass [13]. In the context of trees, Serre gives these "reduced paths" the name"geodesics" (they are shortest paths). We use the word "geodesic" in the context of edge indexed graphs (Section 2.1).

In the sense of other sources, as [14], a "graph" is equivalent to what we are going to call a "combinatorial graph". On the other hand, their definition of a "multi graph" is similar to the following definition of a "graph".

1.1 Graphs, morphisms and subgraphs

A *graph* Γ consists of a set $X = \mathrm{V}\Gamma$, a set $Y = \mathrm{E}\Gamma$ and the *graph maps*

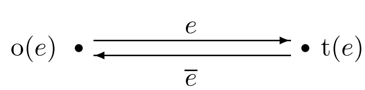
$$\begin{array}{ccccccc} Y & \longrightarrow & X \times X, & e & \longmapsto & (\mathrm{o}(e), \mathrm{t}(e)) \\ Y & \longrightarrow & Y, & e & \longmapsto & \overline{e}, \end{array} \tag{1.1}$$

which satisfy for all $e \in Y$

$$\overline{\overline{e}} = e, \ \overline{e} \neq e \text{ and } \mathrm{o}(e) = \mathrm{t}(\overline{e}). \tag{1.2}$$

The maps o and t are called *point maps*. An element $x \in X$ is called a *vertex* of Γ, an element $e \in Y$ is called an *(oriented) edge* of Γ and $\overline{e}$ is called the *inverse edge*. The map $^-$ is an involution on the set of edges Y. As $^-$ has no fixed

Figure 1.1: A wrong graph diagram



points, its orbits provide a partition of Y into subsets $\{e, \overline{e}\}$ each of which has two edges. Such a set including an edge together with its inverse is called a *geometric edge* (see Figure 1.1) [1] . The vertex $\mathrm{o}(e) = \mathrm{t}(\bar{e})$ is called the *origin* of e and the vertex $\mathrm{t}(e) = \mathrm{o}(\bar{e})$ is called the *terminus* of e. These two vertices are called the *borders* of e. Two vertices are called *adjacent*, if they are the borders of some edge e. An edge b *follows* an edge a if $\mathrm{o}(b) = \mathrm{t}(a)$, and we say then b follows a *at* x for $x = \mathrm{o}(b)$.

We can form for every vertex $x \in \mathrm{V}\Gamma$ the set $\mathrm{St}^{\Gamma}(x) = \{e \in \mathrm{E}\Gamma : \mathrm{o}(e) = x\}$, the *star* at x and write simply $\mathrm{St}(x)$. If $\mathrm{St}(x)$ is finite, then its cardinality is called the *degree* of x, in short $\deg(x)$. Otherwise put $\deg(x) = \infty$.

If $\deg(x) = k$ for all vertices $x \in \mathrm{V}\Gamma$ then Γ is called *regular* or more specifically *k-regular*. If $\deg(x)$ is finite for all vertices, Γ is called *locally finite*. We are only interested in locally finite graphs. A graph is called *finite*, when it has a finite number of edges and vertices.

[1]For an interpretation of diagrams compare page 8. Here both arrows represent the same geometric edge — it is a wrong diagram. Two correct diagrams, representing a graph with one geometric edge and two distinct borders, are given as examples on page 9.

1.1 Definition (Graph morphisms). For two graphs Γ_1 and Γ_2 with corresponding graph maps o,t,$^{-}$ and O,T,$^{\sim}$ respectively, a function

$$F : \begin{cases} \mathrm{V}\Gamma_1 & \longrightarrow & \mathrm{V}\Gamma_2 \\ \mathrm{E}\Gamma_1 & \longrightarrow & \mathrm{E}\Gamma_2 \end{cases}$$

will be denoted simply by $F : \Gamma_1 \to \Gamma_2$. This function is called a *morphism*, if

$$F(\mathrm{o}(e)) = \mathrm{O}(F(e)) \tag{1.3}$$

$$F(\overline{e}) = \widetilde{F(e)} \tag{1.4}$$

holds for all $e \in \mathrm{E}\Gamma_1$.

If there is no danger of confusion, we do not distinguish between the graph maps o,t and $^{-}$ of Γ_1 and Γ_2's maps O,T and $^{\sim}$. The morphism rules read then $F \circ \mathrm{o} = \mathrm{o} \circ F$ and $F \circ ^{-} = ^{-} \circ F$ and we say F is a o - and $^{-}$ - *equivariant* function. Indeed, a function $\Gamma_1 \to \Gamma_2$, which meets (1.3) and (1.4) satisfies also

$$F(\mathrm{t}(e)) = F(\mathrm{o}(\overline{e})) = \mathrm{O}(F(\overline{e})) = \mathrm{O}(\widetilde{F(e)}) = \mathrm{T}(F(e)). \tag{1.5}$$

fro all edges $e \in \mathrm{E}\Gamma_1$. A morphism is therefore a function, which is equivariant under the graph maps (1.1). Conversely $F(\mathrm{t}(e)) = \mathrm{T}(F(e))$ together with (1.4) imply (1.3). To verify the morphism property of a map F between graphs, it is sufficient to show, that F is $^{-}$ - equivariant and equivariant under one of the point maps.

A morphism is called injective, surjective or bijective, if the map on the set of vertices and the map on the set of edges have these properties. A morphism $F : \Gamma \to \Gamma$ from a graph to itself is called an *endomorphism* and we define $\mathrm{End}(\Gamma)$ as the set of all endomorphisms of Γ. A bijective morphism is called an *isomorphism*, a bijective endomorphism is called an *automorphism* and we define $\mathrm{Aut}\,(\Gamma)$ as the set of all automorphisms of Γ.

A morphism α from a graph Γ_1 to a graph Γ_2 is *locally injective*, *locally surjective* respectively *locally bijective* if the restriction

$$\begin{array}{cccc} \alpha_x : & \mathrm{St}^{\Gamma_1}(x) & \longrightarrow & \mathrm{St}^{\Gamma_2}(\alpha(x)) \\ & \bigcap & & \bigcap \\ & \mathrm{E}\Gamma_1 & & \mathrm{E}\Gamma_2 \end{array}$$

of α to the *local map* $\alpha_x = \alpha|_{\mathrm{St}(x)}$ is injective, surjective respectively bijective for each $x \in X$.

1.2 Note (Properties of morphisms). Directly from the rules (1.3) and (1.5) follow some properties of any morphism F, which might be expected for a morphism to hold in the graph model:

- The origin of an edge e is sent under F to the origin of $F(e)$, the terminus of an edge e is sent under F to the terminus of $F(e)$. In particular, F maps adjacent vertices to adjacent vertices.
- If an edge b follows an edge a at the vertex x, then $F(b)$ follows the edge $F(a)$ at $F(x)$.

1.3 Lemma. *The composition of two graph morphisms is a graph morphism.*

Proof. Suppose we have the situation $\Gamma_1 \xrightarrow{F_1} \Gamma_2 \xrightarrow{F_2} \Gamma_3$ for three graphs $\Gamma_1, \Gamma_2, \Gamma_3$ and two morphisms F_1 and F_2. For each $e \in \mathrm{E}\Gamma_1$ we obtain $F_2 \circ F_1(\mathrm{o}(e)) = F_2(\mathrm{o}(F_1(e))) = \mathrm{o}(F_2 \circ F_1(e))$ and $F_2 \circ F_1(\overline{e}) = F_2(\overline{F_1(e)}) = \overline{F_2 \circ F_1(e)}$ in accordance with (1.3) and (1.4). $\square$

1.4 Definition (Subgraphs). A *subgraph* Γ' of a graph Γ consists of a set of vertices $\mathrm{V}\Gamma' \subset \mathrm{V}\Gamma$ and a set of edges $\mathrm{E}\Gamma' \subset \mathrm{E}\Gamma$ satisfying

$$\begin{aligned} \mathrm{o}(\mathrm{E}\Gamma') &\subset \mathrm{V}\Gamma' \\ \overline{\mathrm{E}\Gamma'} &= \mathrm{E}\Gamma'. \end{aligned} \tag{1.6}$$

In this case we write $\Gamma' < \Gamma$.

A subgraph may be *generated* by a set of vertices $X \subset \mathrm{V}\Gamma$. We denote by $\langle X \rangle$ the subgraph of Γ with vertices X and edges $\{e \in \mathrm{E}\Gamma : \mathrm{o}(e), \mathrm{t}(e) \in X\}$.

A subgraph Γ' of a graph Γ is a graph. We take as graph maps (1.1) for Γ' the restrictions of the graph maps for Γ. Γ' is closed under these maps, since for all $e \in \mathrm{E}\Gamma'$ we get $\overline{e} \in \mathrm{E}\Gamma'$, $\mathrm{o}(e) \in \mathrm{VE}\Gamma'$ and $\mathrm{t}(e) = \mathrm{o}(\overline{e}) \in \mathrm{V}\Gamma'$. The restrictions of o, t and $^{-}$ to $\mathrm{E}\Gamma'$ meet then the conditions (1.2) for all $e \in \mathrm{E}\Gamma'$, as they do more generally for $e \in \mathrm{E}\Gamma$.

1.5 Example (Subgraphs of a graph Γ).

- Any set $\mathrm{V}\Gamma' \subset \mathrm{V}\Gamma$ together with $\mathrm{E}\Gamma' = \emptyset$ forms a subgraph of Γ.
- A union $\mathrm{E}\Gamma'$ of geometric edges and a set $\mathrm{V}\Gamma'$ containing the borders of all that edges form a subgraph Γ' of Γ.

For a morphism $F : \Gamma_1 \to \Gamma_2$ we define $\mathrm{V}(F\Gamma_1) := F(\mathrm{V}\Gamma_1)$ and $\mathrm{E}(F\Gamma_1) := F(\mathrm{E}\Gamma_1)$. $F\Gamma_1$ is then a graph in a natural way: By the morphism rules 1.3 and 1.4 we get $\mathrm{O}(F(e)) = F(\mathrm{o}(e)) \in F(\mathrm{V}\Gamma_1) = \mathrm{V}(F\Gamma_1)$ and $\widetilde{F(e)} = F(\overline{e}) \in F(\mathrm{E}\Gamma_1) = \mathrm{E}(F\Gamma_1)$ for all edges $F(e) \in \mathrm{E}(F\Gamma_1)$.

It is also easy to verify, that both the intersection and the union of two subgraphs are a subgraph.

1.2 Oriented graphs

An *orientation* of a graph Γ is a subset Y_+ of $Y = \mathrm{E}\Gamma$ such that Y is the disjoint union of Y_+ and $\overline{Y_+}$.

1.6 Lemma. *An orientation exists for all graphs.*

Proof. Consider the partition of $\mathrm{E}\Gamma$ in orbits of $\bar{\ }$. Each block of the partition has two edges $\{e, \overline{e}\}$. Using the axiom of choice we can form Y_+ as a set consisting of one element out of each block.

Suppose some edge e is not in Y_+. Then $\overline{e} \in Y_+$, and hence $e \in \overline{Y_+}$. That shows $Y = Y_+ \cup \overline{Y_+}$. If $e \in Y_+$ and $e \in \overline{Y_+}$, then $e, \overline{e} \in Y_+$. But this is a contradiction to the choice of Y_+. Together we conclude $Y = Y_+ \sqcup \overline{Y_+}$. ☐

1.7 Definition (Oriented graphs). An *oriented graph* is defined by a set X of vertices, a set Y_+ of *positive edges*, a map $Y_+ \to X \times X$, $e \mapsto (\mathrm{o}(e), \mathrm{t}(e))$ and a bijection α from Y_+ to a disjoint copy $\overline{Y_+}$ of Y_+. We can extend the graph maps to the whole set of edges $Y_+ \cup \overline{Y_+}$ by

$$\overline{e} := \alpha(e) \qquad \text{for all } e \in Y_+$$

$$\left.\begin{aligned} \overline{e} &:= \alpha^{-1}(e) \\ \mathrm{o}(e) &:= \mathrm{t}(\overline{e}) \\ \mathrm{t}(e) &:= \mathrm{o}(\overline{e}) \end{aligned}\right\} \qquad \text{for all } e \in \overline{Y_+}$$

and obtain a graph by Lemma 1.8. A graph morphism between oriented graphs is called *orientation preserving*, if all positive edges are mapped to positive edges.

1.8 Lemma. *An oriented graph is a graph with vertices X and edges $Y = Y_+ \sqcup \overline{Y_+}$. The set of positive edges Y_+ is an orientation for this graph.*

Proof. We have to check (1.2). The graph map $\bar{\ }$, as introduced above, is clearly a bijection on the edges $Y = Y_+ \sqcup \overline{Y_+}$ and fulfills $\overline{\overline{e}} = e$ for all edges, since if $e \in Y_+$ then $\overline{\overline{e}} = \overline{\alpha(e)} = \alpha^{-1}\alpha(e) = e$, while for $e \in \overline{Y_+}$ we have $\overline{\overline{e}} = \overline{\alpha^{-1}(e)} = \alpha\alpha^{-1}(e) = e$. In the first case $e = \overline{e}$ gives the contradiction $e = \overline{\overline{e}} = \alpha(\overline{e}) \in \overline{Y_+}$, in the second case the contradiction $\overline{e} = \alpha^{-1}(e) \in \overline{Y_+}$.

By definition, $\mathrm{o}(a) = \mathrm{t}(\overline{a})$ for all $a \in \overline{Y_+}$. As α is a bijection from Y_+ to $\overline{Y_+}$, $\mathrm{t}(b) = \mathrm{o}(\overline{b})$ for all $b \in \overline{Y_+}$ implies $\mathrm{o}(c) = \mathrm{t}(\overline{c})$ for all $c \in Y_+$ and therefore $\mathrm{o}(e) = \mathrm{t}(\overline{e})$ for all edges $e \in Y$.

$\overline{Y_+}$ has been chosen disjoint to Y_+, hence Y_+ is an orientation for the graph.

□

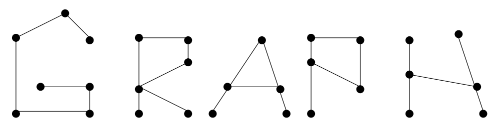
1.9 Proposition. *An oriented graph is defined (up to a trivial orientation preserving automorphism) by a set X of vertices, a set Y_+ of positive edges and a map $Y_+ \to X \times X$, $e \mapsto (\mathrm{o}(e), \mathrm{t}(e))$.*

Proof. We may choose bijections $\alpha : Y_+ \to \overline{Y_+}$ and $\beta : Y_+ \to \overline{Y_+}$ to a disjoint copy of Y_+. Then with the bijection α and Lemma 1.8, a graph Γ_α with vertices X, edges $Y = Y_+ \sqcup \overline{Y_+}$ and orientation Y_+ is defined. The graph maps of Γ_α shall be written as o, t and $\bar{\ }$. In the same way one obtains from β a graph Γ_β with graph maps O, T and $\tilde{\ }$. We can now write a function F (the trivial orientation preserving automorphism) from Γ_α to Γ_β as

$$F : \begin{cases} Y_+ \longrightarrow Y_+, & e \mapsto e \\ \overline{Y_+} \longrightarrow \overline{Y_+}, & e \mapsto \tilde{\overline{e}} \\ X \longrightarrow X, & x \mapsto x. \end{cases}$$

The proposition is proved, if we can show, that F is a morphism, since bijectivity follows directly from that of α and β and the orientation is preserved by construction. Suppose $e \in Y_+$ and $\overline{e} \in \overline{Y_+}$. Then $F(\overline{e}) = \tilde{\overline{\overline{e}}} = \tilde{e} = \widetilde{F(e)}$. If $e \in \overline{Y_+}$ and $\overline{e} \in Y_+$, then $F(\overline{e}) = \overline{e} = \tilde{\tilde{\overline{e}}} = \widetilde{F(e)}$.
We have to check also the origin function. For $e \in Y_+$ we get $F(\mathrm{o}(e)) = \mathrm{o}(e) = \mathrm{O}(e) = \mathrm{O}F(e)$, where the middle equality holds, because o and O coincide on Y_+. If $e \in \overline{Y_+}$, we get $F(\mathrm{o}(e)) = \mathrm{o}(e) = \mathrm{t}(\overline{e}) = \mathrm{T}(\overline{e}) = \mathrm{T}(F(\overline{e})) = \mathrm{T}(\widetilde{F(e)}) = \mathrm{O}(F(e))$ using $F \circ \bar{\ } = \tilde{\ } \circ F$ and the coincidence of T and t on Y_+. □

Figure 1.2: A first graph diagram



1.3 Diagrams

Graphs are represented pictorially in accordance with the following convention: a point marked on the diagram corresponds to a vertex of the graph, a line joining two marked points corresponds to a geometric edge.

We draw an arrow instead of a line in diagrams of graphs, if we want to refer to an edge rather than to a geometrical edge with some label (cf. Figure 1.4). In oriented graphs, arrows are used to identify origin and terminus (direction) of positive edges (cf. Figure 2.9 on page 36).

Sometimes, when drawing large graphs, we avoid drawing vertices (i.e. we remove them after drawing the lines), as for example on page 16 or on page 72. Not alway is it then possible to recover the structure of a graph from the diagram, still it will remain intuitively clear in the indicated examples.

1.10 Example (Graphs and diagrams).

- The graph having one vertex x and two edges e,$\overline{e}$ is represented by each of the diagrams in Figure 1.3.

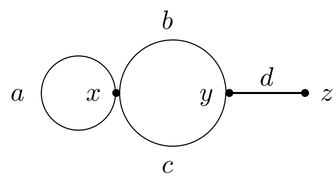
Figure 1.3: Diagrams with one point and one line

- The graph having two edges $e, \overline{e}$, with two distinct borders x, y as only vertices is represented by each of the diagrams in Figure 1.4. The first diagram does not specify if x or y is the origin of e. The second diagram does.

Figure 1.4: Diagrams with two points joined by a line

- The diagram in Figure 1.5 represents a graph with three vertices x, y, z and eight edges $a, \overline{a}, b, \overline{b}, c, \overline{c}, d, \overline{d}$. The diagram specifies $\mathrm{o}(a) = \mathrm{t}(a) =$

Figure 1.5: Another diagram



$\mathrm{o}(\overline{a}) = \mathrm{t}(\overline{a}) = x$. a, b, c, d have borders x, x; x, y; x, y and y, z respectively.

1.4 Special morphisms and their features

1.4.1 Paths — distance and connection

Paths are fundamental constituents in the language of graphs. Several operations will be defined for paths and some path-related attributes can be given to graphs.

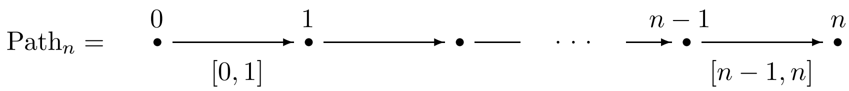
1.11 Definition (Paths). Let $n > 0$ be an integer. The oriented graph path_n has $n+1$ vertices $0, 1, \ldots, n$. The orientation is given by the n edges $[i, i+1]$, $0 \leq i < n$, where $\mathrm{o}([i, i+1]) = i$ and $\mathrm{t}([i, i+1]) = i+1$ (see Figure 1.6). A morphism p from Path_n to a graph Γ is called a *path*, n is called the *length* of p and will be denoted as $\mathrm{len}(p)$.

The positive edges $[i, i+1]$ of Path_n map under p to the sequence of n edges

$$p([0,1]), \ldots, p([n-1, n]),$$

called the *edge sequence*. These edges satisfy $\mathrm{t}(p[i, i+1]) = \mathrm{o}(p[i+1, 1+2])$,

Figure 1.6: The graph path_n



because p is a morphism. The sequence

$$p(0), \ldots, p(n)$$

is called *vertex sequence*. Two consecutive vertices of the vertex sequence of a path are adjacent.

For $n > 0$, a sequence of n edges $e_1, \ldots, e_n$ in a graph Γ satisfying $\mathrm{t}(e_i) = \mathrm{o}(e_{i+1})$ for $1 \leq i < n$ defines a graph morphism p from Path_n to Γ by putting $F([i, i+1]) := e_{i+1}$ and extending this function by the morphism rules. Hence a path p for $\mathrm{len}(p) = n > 0$ may be defined as

$$p = (e_1, \ldots, e_n) \in (\mathrm{E}\Gamma)^n \tag{1.7}$$

with $\mathrm{t}(e_i) = \mathrm{o}(e_{i+1})$ for all $1 \leq i \leq n-1$. It will be convenient to change between the preceding two definitions of paths.

For completeness one should mention also paths of length zero. The graph Path_0 consists of a single vertex and no edges. A path of length zero is represented by a vertex in a graph and has an empty edge sequence. The following operations defined for paths of positive lengths can be extended to paths of length zero without ambiguity and will be used frequently.

For a graph Γ and a path $p = (e_1, \ldots, e_n)$, we assign to p the *origin* $\mathrm{o}(p)$ and the *terminus* $\mathrm{t}(p)$

$$\mathrm{o}(p) = \mathrm{o}(e_1)$$

$$\mathrm{t}(p) = \mathrm{t}(e_n)$$

and say, that p *joins* $\mathrm{o}(p)$ with $\mathrm{t}(p)$, or that p is a path *from* $\mathrm{o}(p)$ *to* $\mathrm{t}(p)$. If F is a morphism from Γ to another graph, we know from Lemma 1.3, that $F \circ p$

is path in $F\Gamma$ and it is easy to verify, that

$$\begin{aligned} F\mathrm{o}(p) &= \mathrm{o}F(p) \\ F\mathrm{t}(p) &= \mathrm{t}F(p). \end{aligned} \tag{1.8}$$

We assign to p the *inverse path*

$$\overline{p} = \overline{e_n}, \ldots, \overline{e_1}.$$

This assignment indeed defines a path, since $\mathrm{t}(\overline{e_{i+1}}) = \mathrm{o}(e_{i+1}) = \mathrm{t}(e_i) = \mathrm{o}(\overline{e_i})$ holds for all $1 \leq i < n-1$. Obviously

$$\begin{aligned} \overline{\overline{p}} &= p \\ \mathrm{o}(\overline{p}) &= \mathrm{t}(p). \end{aligned} \tag{1.9}$$

For morphisms F from Γ to another graph we get $\overline{F(p)} = (\overline{Fe_n}, \ldots, \overline{Fe_1}) = (F\overline{e_n}, \ldots, F\overline{e_1}) = F(\overline{p})$ and hence

$$F(\overline{p}) = \overline{F(p)}. \tag{1.10}$$

For two paths $p = (e_1, \ldots, e_n)$ and $q = (a_1, \ldots, a_m)$ the *composition* is defined as the edge sequence

$$pq = (e_1, \ldots, e_n, a_1, \ldots, a_m).$$

If $\mathrm{t}(p) = \mathrm{o}(q)$ holds, then of course pq is a path ($\mathrm{t}(p) = \mathrm{o}(q)$ is equivalent to $\mathrm{t}(e_n) = \mathrm{o}(a_1)$). A path $p = (e_1, \ldots, e_n)$ is called *closed* if and only if the composition pp is a path. Note that this is equivalent to $\mathrm{t}(e_n) = \mathrm{o}(e_1)$.

A path $p = e_1, \ldots, e_n$ has *first edge* e_1 and *last edge* e_n. p *leads* from its first edge to its last one. For two paths $p = (e_1, \ldots, e_n)$ and $q = (a_1, \ldots, a_m)$, the *concatenation* of p and q is defined as the edge sequence

$$p \wedge q = (e_1, \ldots, e_{n-1}, a_1, \ldots, a_m).$$

Clearly, the concatenation of two paths p and q is a path whenever $e_n = a_1$ — and it will only be used in that case.

1.12 Definition (Vertex distance). Let Γ be a graph. We define a *distance* between vertices by

$$\mathrm{d}(x, y) = \inf \left\{ \mathrm{len}(p) \;\middle|\; \begin{array}{r} p \text{ is a path} \\ \text{joining } x \text{ with } y \end{array} \right\}$$

(note that the infimum of an empty set is $+\infty$). A graph Γ is called *connected* if and only if each pair of vertices is joined by a path. The *connected component* $\mathcal{C}(x)$ of a vertex x in a graph Γ is the subgraph $\langle\{y \in \mathrm{V}\Gamma : \mathrm{d}(x, y) < \infty\}\rangle < \Gamma$.

1.13 Lemma. *If Γ is a connected graph, then the distance d is a metric on the vertex set $\mathrm{V}\Gamma$.*

Proof. Trivially $\mathrm{d}(x, y) = 0 \Leftrightarrow x = y$. If p joins x with y, then $\overline{p}$ joins y with x. As $\mathrm{len}(p) = \mathrm{len}(\overline{p})$ we obtain $\mathrm{d}(x, y) = \mathrm{d}(y, x)$. If p is a path joining x with y and q is a path joining y to z, then pq joins x with z. If we chose p and q minimal in length, then we get $\mathrm{d}(x, z) \leq \mathrm{len}(pq) = \mathrm{len}(p) + \mathrm{len}(q) = \mathrm{d}(x, y) + \mathrm{d}(y, z)$. This shows transitivity. □

1.14 Lemma. *If A is a connected graph, then $F(A)$ is connected for all morphisms F on A.*

Proof. Each two vertices of $F(A)$ are of the form Fx, Fy for two vertices $x, y \in \mathrm{V}A$. There is a path p from x to y in A by connection. The composition $F \circ p$ is a path in $F(A)$ from Fx to Fy. □

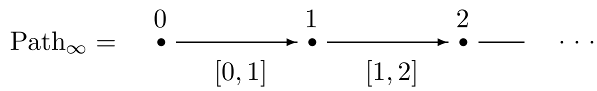
1.15 Lemma (Basic properties of connected components).

- *A connected component is a connected subgraph,*
- *a graph is the disjoint union of its connected components,*
- *a graph is connected if and only if it has exactly one connected component.*

Proof. A connected component is connected because all its vertices are mutually within finite distance hence linked by a path. The connected components of a graph trivially cover the same graph. If two components have non-empty intersection, the intersection as a subgraph has a vertex, which lies in both components. The components coincide. As for the last statement, the first two statements give one direction. Conversely, if a graph is connected, then clearly all connected components coincide. □

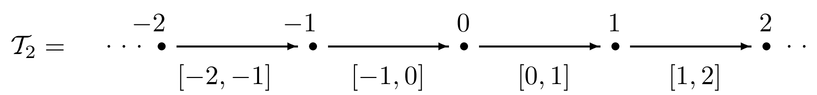
We will need to use non-finite paths. There is the one-sided version path_∞, which has vertices $0, 1, 2, \ldots$ The orientation is given by the edges $[i, i+1]$, $i \in \mathbb{N}_0$, where $\mathrm{o}([i, i+1]) = i$ and $\mathrm{t}([i, i+1]) = i + 1$ (compare Figure 1.7).

Figure 1.7: The graph path_∞



A morphism p from Path_∞ to a graph Γ is called an *infinite path* or a *ray*. There is the two-sided version $\mathcal{T}_2$, with vertices $\mathbb{Z}$. The orientation is given by the edges $[i, i+1]$, $i \in \mathbb{Z}$, where $\text{o}([i, i+1]) = i$ and $\text{t}([i, i+1]) = i+1$ (cf. Figure 1.8)[2] . A morphism p from $\mathcal{T}_2$ to a graph Γ is called a *bi-infinite*

Figure 1.8: The graph $\mathcal{T}_2$



path. As in the case of finite paths we can identify a ray with its edge sequence $p([0,1]), p([1,2]), \ldots$. Similarly we identify a bi-infinite path p with the edge sequence $(\ldots, p[-2,-1], p[-1,0], p[0,1], p[1,2], \ldots)$[3] .

1.16 Definition (Segments of paths). Given a path of length $n \geq 1$ ($I = \{1, \ldots, n\}$), a ray ($I = \mathbb{N}$) or a bi-infinite path ($I = \mathbb{Z}$) by its edge sequence $p = \{e_i\}_{i \in I}$, a *segment* of p of length $l > 0$ is a path of the form $e_k, \ldots, e_{k+l-1}$ with $\{k, \ldots, k+l-1\} \subset I$.

1.17 Definition (Positive paths). A *positive path* in an oriented graph Γ with orientation $Y_+ \subset \text{E}\Gamma$ is a path, which has an edge sequence including only positive edges, i.e. edges of Y_+. A path p is called a *closed positive path*, if p is a positive path and p is closed. We define

$$\mathcal{P}(\Gamma)$$

as the set of *positive bi-infinite* paths in Γ.

[2] $\mathcal{T}_2$ is also called the 2-regular tree

[3] A $\mathbb{Z}$-sequence of edges is a function from $\mathbb{Z}$ to some edge set, not only an ordering on a countable edge set.

1.4.2 Circuits — combinatorial graphs and trees

Let $n > 0$ be an integer. Consider the oriented graph circ_n with n vertices $0, \dots, n-1$ and n positive edges $[0,1], \dots, [n-1, 0]$, where $\mathrm{o}([i, i+1]) = i$ and $\mathrm{t}([i, i+1]) = i+1$ ($i \bmod n$). There is a diagram of circ_n in Figure 1.9.

$\mathrm{circ}_n :=$

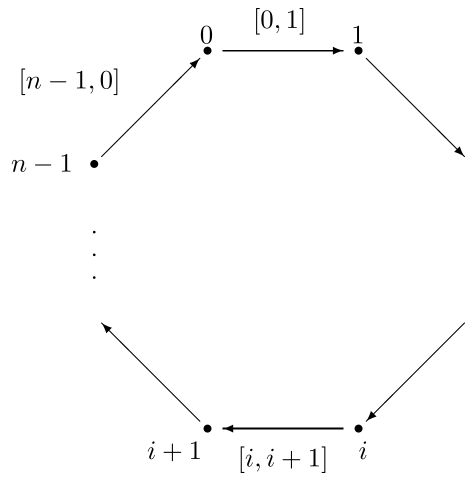


Figure 1.9: The oriented graph circ_n

Isomorphic images of these graphs are called *circuits*. The graph circ_1 is also called a *loop* (see Figure 1.3 on page 8).

1.18 Definition (Combinatorial graphs). A graph is called *combinatorial* if it has no circuits of length ≤ 2.

Serre argues in [6] Chapter 2, that the structure of a combinatorial graph is fully described, if one knows about adjacency of the vertices, because each ordered pair of adjacent vertices is joined by a unique edge. Therefore we can label the edges conveniently by their border points. If x and y are adjacent, then (x, y) denotes the unique edge from x to y, and (y, x) the unique edge from y to x. In particular $(y, x) = \overline{(x, y)}$.

1.19 Lemma. *A function $F : \mathrm{V}A \to \mathrm{V}B$ from the vertices $\mathrm{V}A$ of a graph A to the vertices $\mathrm{V}B$ of a combinatorial graph B mapping adjacent vertices to adjacent vertices extends to a unique morphism from A to B.*

Proof. For each edge $e \in A$, the vertices $F(\mathrm{o}(e))$ and $F(\mathrm{t}(e))$ are adjacent in B. They are linked by the unique edge $(F(\mathrm{o}(e)), F(\mathrm{t}(e)))$. So there is no other way than to put $F(e) := (F(\mathrm{o}(e)), F(\mathrm{t}(e))$ in order to comply with (1.3) and (1.5). The rule (1.4) is verified by $\overline{F(e)} = \overline{(F(\mathrm{o}(e)), F(\mathrm{t}(e)))} = (F(\mathrm{t}(e)), F(\mathrm{o}(e))) = (F(\mathrm{o}(\overline{e})), F(\mathrm{t}(\overline{e}))) = F(\overline{e})$. ☐

We use this property to simplify notation. The vertex sequence of a path determines by Lemma 1.19 the entire path. This allows us to identify a path with its vertex sequence, provided, that it is a path in a combinatorial graph. In a general graph this reduction may lead to ambiguity. A path defined by an edge e of a loop for instance had a vertex sequence $(\mathrm{o}(e), \mathrm{t}(e))$, the same sequence as the path $\overline{e}$ has.

1.20 Definition (Trees). A *tree* is a non-empty connected graph without circuits.

1.21 Example (Combinatorial graphs).

- Every tree is a combinatorial graph.
- Figure 1.2 on page 8 shows a diagram of a graph. The connected components (from left to right) are a tree, a combinatorial graph, a combinatorial graph, a combinatorial graph and a tree respectively.

A segment of the form $(e, \overline{e})$ in a path is called a *reversal*. A *reduced path* in a graph Γ is a path p, that has an edge sequence without reversals. An infinite path is called reduced, if each of its segments of length two is reduced. We define

$$\begin{aligned} \mathcal{R}_\infty(\Gamma) &:= \{ \text{ reduced rays in } \Gamma \} \\ \mathcal{R}(\Gamma) &:= \{ \text{ reduced bi-infinite paths is } \Gamma \} \end{aligned}$$

The involution $^-$ maps reduced paths to reduced paths because $b = \overline{a} \Rightarrow \overline{\overline{a}} = \overline{b}$. A path p is a *closed reduced path* if and only if the composition pp is a reduced path [4] .

1.22 Proposition. *Two vertices in a tree are joined by a unique reduced path. This reduced path is an injective path.*

[4] The composition of two reduced paths does not need to be a reduced path, as the example $a\overline{a}$ shows for edges a.

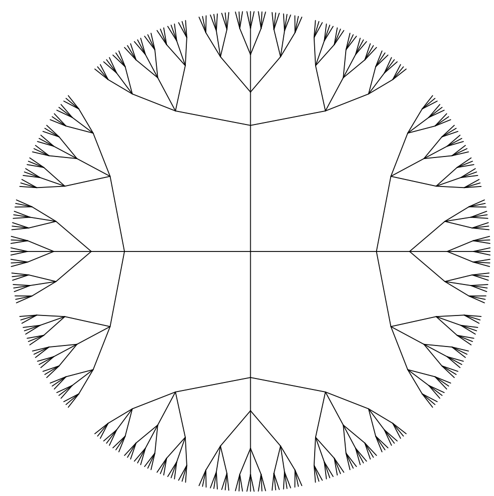
Proof. A proof shall not be given here. There is a very concise proof in [6]. □

This fundamental property will avail notation: $[x, y]$ is defined as the unique reduced path from x to y for vertices x, y of a tree.

1.23 Definition. A k-regular tree ($k \geq 2$) will be denoted as $\mathcal{T}_k$.

Two k-regular trees $\mathcal{S}$ and $\mathcal{T}$ are isomorphic. A morphism from $\mathcal{S}$ to $\mathcal{T}$ can be chosen locally bijective. This morphism is then an isomorphism (cf. Section 1.4.3). In Figure 1.10 a part of $\mathcal{T}_4$ is drawn.

Figure 1.10: 4-regular tree



The following general statement will prove useful: If Γ is a graph, x, z are vertices of Γ and p is a path from x to z, then $\mathrm{d}(x, z) = \mathrm{len}(p)$ implies p is reduced (otherwise there was a shorter path from x to z, compare Definition 1.12). The definition of the vertex distance can therefore be restated as

$$\mathrm{d}(x, y) = \inf \{\mathrm{len}(p) : \ p \text{ is a reduced path joining } x \text{ with } y\} . \qquad (1.11)$$

1.24 Lemma. *Suppose $\mathcal{T}$ is a tree, $x, z \in \mathrm{V}\mathcal{T}$ and p is a path from x to z. Then*

- $\mathrm{d}(x, z) = \mathrm{len}(p)$ *if and only if p is reduced,*
- $\mathrm{d}(x, z) = \mathrm{len}(p) \bmod 2$.

Proof. Above we saw, that $\mathrm{d}(x,z) = \mathrm{len}(p)$ implies that p is reduced in a general graph. In case of a tree we know about uniqueness of reduced paths, hence

$$\begin{aligned} \mathrm{d}(x,z) &\overset{(1.11)}{=} \inf\{\mathrm{len}(q) : \ q \text{ is a reduced path joining } x \text{ with } z\ \} \\ &\overset{\text{Proposition 1.22}}{=} \inf\{\mathrm{len}(q) : q = [x,z]\} \\ &= \mathrm{len}[x,z]. \end{aligned}$$

If p is reduced, then $p = [x,z]$, hence $\mathrm{d}(x,z) = \mathrm{len}(p)$.

For the second assertion, one may remove all reversals from p ending up with the reduced path $[x,z]$. Each reversal has length two, so p has been shortened by an even number. □

1.25 Lemma. *Suppose $\mathcal{T}$ is a tree and $x, y, x \in \mathrm{V}\mathcal{T}$. Equivalent are*

a) $[x,y][y,z]$ *is reduced,*

b) $[x,y][y,z] = [x,z]$,

c) $y \in [x,z]$,

d) $\mathrm{d}(x,y) + \mathrm{d}(y,z) = \mathrm{d}(x,z)$.

Proof. With Lemma 1.24, these properties follow easily. □

1.26 Lemma. *If $\mathcal{T}$ is a tree, $e \in \mathrm{E}\mathcal{T}$ and $x \in \mathrm{V}\mathcal{T}$, then* $\mathrm{d}(x, \mathrm{o}(e)) \neq \mathrm{d}(x, \mathrm{t}(e))$.

Proof. We choose $p = [x, \mathrm{o}(e)]$ and $q = [x, \mathrm{t}(e)]$. $\overline{p}q$ is a path from $\mathrm{o}(e)$ to $\mathrm{t}(e)$. $\mathrm{d}(\mathrm{o}(\overline{p}q), \mathrm{t}(\overline{p}q)) = \mathrm{d}(\mathrm{o}(e), \mathrm{t}(e)) = 1$, since e is reduced. $\mathrm{len}(\overline{p}q) = \mathrm{len}(p) + \mathrm{len}(q) = 2 \cdot \mathrm{len}(p)$ if we suppose that $\mathrm{d}(x, \mathrm{o}(e)) = \mathrm{d}(x, \mathrm{t}(e))$. This is a contradiction to Lemma 1.24's second assertion. □

1.4.3 Corollaries of local behavior

Sometimes it is possible to derive a global property of a morphism from its local behavior[5] . Suppose A and B are graphs and ϕ is a morphism from A to B:

a) If $A \neq \emptyset$, B is connected and ϕ is locally surjective, then ϕ is surjective.

b) If A is connected, B has no circuits and ϕ is locally injective, then ϕ is injective.

[5] This nice enumeration is copied from H. Bass [7]. There is a proof below for "easy verification".

c) If $A \neq \emptyset$ is connected, B is a tree and ϕ is locally bijective, then ϕ is bijective.

a) If B has no edges, then by connection it consists of a single vertex. This case is trivial. Let $e \in \mathrm{E}B$ and $x \in \mathrm{V}(\phi A)$. By connection of B there is a path p in B from x to $\mathrm{o}(e)$. Since x is a vertex of ϕA, it follows inductively that the edge sequence of p is in $\mathrm{E}(\phi(A))$, using only local surjectivity. Hence $\mathrm{o}(e) = \mathrm{t}(p) \in \mathrm{V}(\phi A)$ and therefore $e \in \mathrm{E}(\phi A)$ by local surjectivity again. This shows $\mathrm{E}(\phi A) = \mathrm{E}B$, hence $\phi A = B$ by connection of B.

b) We show first, that $\phi : \mathrm{V}A \to \mathrm{V}B$ is injective. Suppose x, y are vertices of A. We can choose by connection a reduced path r from x to y. As ϕ is locally injective, it maps reduced paths to reduced paths, hence $\phi \circ r$ is a reduced path in the connected component $\mathcal{C}(\phi(x)) < B$ of $\phi(x)$. This component is connected, hence a tree, and we can write $\phi(r) = [\phi(x), \phi(y)]$. If $\phi(x) = \phi(y)$, then $\mathrm{len}(\phi(r)) = 0$, hence $\mathrm{len}(r) = 0$ hence $x = y$. For two edges a, b we may assume $\phi(a) = \phi(b)$. Then by injectivity on vertices one has $\mathrm{o}(a) = \mathrm{o}(b)$, hence by local injectivity $a = b$.

c) This is the logical conjunction of case a) and case b).

1.5 Group actions on graphs

1.5.1 Quotients of graphs

In the first section of this chapter we saw already, that a composition of morphisms is a morphism (cf. Lemma 1.3). Moreover, if F is an isomorphism from a graph Γ_1 to a graph Γ_2, then the inverse function F^{-1} from Γ_2 to Γ_1 is also an isomorphism. For every edge $d \in \mathrm{E}\Gamma_2$ and $e = F^{-1}(d)$ one verifies the equations

$$F^{-1}(\tilde{d}) = F^{-1}(\widetilde{F(e)}) = F^{-1}F(\overline{e}) = \overline{e} = \overline{F^{-1}(d)}$$
$$F^{-1}(\mathrm{O}(d)) = F^{-1}(\mathrm{O}(F(e))) = F^{-1}F(\mathrm{o}(e)) = \mathrm{o}(e) = \mathrm{o}(F^{-1}(d)).$$

We are mostly interested in automorphisms of a graph Γ. So we take $\Gamma_1 = \Gamma_2$. From the above discussion one can see that the set $\mathrm{Aut}\,(\Gamma)$ of automorphisms of a graph Γ is a group under composition. We call it the *automorphism group of* Γ. An automorphism F of a graph Γ is called an *inversion*, if there is an edge $e \in \mathrm{E}\Gamma$ such that $F(e) = \overline{e}$.

1.27 Definition (Quotient graph). If a group G acts on an graph Γ by automorphisms and without inversions, we define the *quotient graph*

$$\mathrm{G}\backslash\Gamma.$$

This graph has vertices $\mathrm{V}(\mathrm{G}\backslash\Gamma) := \mathrm{G}\backslash\mathrm{V}\Gamma = \{\mathrm{G}x : x \in \mathrm{V}\Gamma\}$ and edges $\mathrm{E}(\mathrm{G}\backslash\Gamma) := \mathrm{G}\backslash\mathrm{E}\Gamma = \{\mathrm{G}e : e \in \mathrm{E}\Gamma\}$ (cf. Appendix B).

$\mathrm{G}\backslash\Gamma$ is indeed a graph. Due to the equivariance of a morphism under graph maps, the equations

$$\begin{aligned} \mathrm{o}(\mathrm{G}e) &= \mathrm{Go}(e) \\ \mathrm{t}(\mathrm{G}e) &= \mathrm{Gt}(e) \\ \overline{\mathrm{G}e} &= \mathrm{G}\overline{e} \end{aligned} \tag{1.12}$$

hold and can be used to define graph maps (1.1) for the quotient graph. These maps already obey the graph rules (1.2), since by repeated application of (1.12) we get $\overline{\overline{\mathrm{G}e}} = \mathrm{G}\overline{\overline{e}} = \mathrm{G}e$ and $\mathrm{o}(\mathrm{G}e) = \mathrm{Go}(e) = \mathrm{Gt}(\overline{e}) = \mathrm{t}(\mathrm{G}\overline{e}) = \mathrm{t}(\overline{\mathrm{G}e})$, except possibly the rule $\overline{\mathrm{G}e} \neq \mathrm{G}e$.

The condition on inversions must be taken into account at this point. If an element $g \in \mathrm{G}$ is an inversion, say $g(e) = \overline{e}$ for $e \in \mathrm{E}\Gamma$, then $\overline{\mathrm{G}e} = \overline{\mathrm{G}g(e)} = \overline{\mathrm{G}\overline{e}} = \mathrm{G}e$ violates (1.2). Conversely if there is an edge $\mathrm{G}e \in \mathrm{E}(\mathrm{G}\backslash\Gamma)$ such that $\overline{\mathrm{G}e} = \mathrm{G}e$, then $\mathrm{G}\overline{e} = \mathrm{G}e$ implies the existence of an group element $g \in \mathrm{G}$ with $\overline{e} = g(e)$, thus there is an inversion.

1.28 Definition (Quotient morphism). If a group G acts on an graph Γ by automorphisms and without inversions, we define a *quotient map* $\pi : \Gamma \to \mathrm{G}\backslash\Gamma$ in addition to the quotient graph $\mathrm{G}\backslash\Gamma$. π is specified by $\pi(x) = Gx$ and $\pi(e) = Ge$ for vertices x and edges e. The above equations (1.12) then translate to

$$\begin{aligned} \mathrm{o}(\pi(e)) &= \pi(\mathrm{o}(e)) \\ \overline{\pi(e)} &= \pi(\overline{e}) \end{aligned} \tag{1.13}$$

and prove, that the quotient map is a morphism of graphs.

Suppose a group acts without inversion on a graph A with quotient graph B and quotient map $\pi : A \to B$. Then for paths q in B and vertices x in A, a path p in A is called a *lift of q at x*, if the equations

$$\begin{aligned} \mathrm{o}(p) &= x \\ \pi \circ p &= q \end{aligned}$$

hold. If the first equation is not important, then p is called simply a *lift* of q.

1.29 Lemma. *Suppose a group $G < \mathrm{Aut}\,(A)$ acts without inversions on a graph A with quotient graph $B = G\backslash A$ and quotient morphism π. Then for all paths q in B and all vertices $x \in \pi^{-1}(\mathrm{o}(q))$ there exists a lift q of p at x.*

Proof. For paths of length one, say $q = e \in \mathrm{E}B$ and $x \in \pi^{-1}\{\mathrm{o}(e)\}$, one can choose by surjectivity of the quotient map an edge $a \in \pi^{-1}\{e\}$. Since $\pi(\mathrm{o}(a)) = \mathrm{o}(\pi(a)) = \mathrm{o}(e) = \pi(x)$ there is a group element g with $g(\mathrm{o}(a)) = x$. Then the path $p = ga$ satisfies the required properties. For larger paths, one can augment inductively a shorter path by composition of an edge to the terminus of the shorter one. □

1.5.2 Isometries of a tree

An endomorphism F of a graph Γ is an *isometry* if d is *invariant* under F. We write $\mathrm{Is}\,(\Gamma)$ for the set of isometries of Γ, i.e.

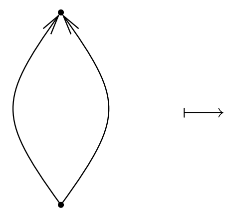
$$\mathrm{Is}\,(\Gamma) = \{F \in \mathrm{End}(\Gamma) : \mathrm{d}\,(F\,(x)\,, F\,(y)) = \mathrm{d}\,(x, y) \text{ for all } x, y \in \mathrm{V}\Gamma\}\,.$$

We say that $\mathrm{Is}\,(\Gamma)$ is the *isometry group* of the graph Γ, if $\mathrm{Is}\,(\Gamma)$ is a group, as for example in the case $\mathrm{Is}\,(\Gamma) = \mathrm{Aut}\,(\Gamma)$.

Clearly $\mathrm{Aut}\,(\Gamma) \subset \mathrm{Is}\,(\Gamma)$. For $x, y \in \mathrm{V}\Gamma$ we can choose a path $p : \mathrm{path}_n \to \Gamma$ with $\mathrm{o}(p) = x, \mathrm{t}(p) = y$. Then $F \circ p$ is also a path with $\mathrm{o}(F \circ p) = F(x), \mathrm{t}(F \circ p) = F(y)$, i.e. $\mathrm{d}(F(x), F(y)) \leq \mathrm{d}(x, y)$. If we argue with F^{-1} instead of F, we obtain $\mathrm{d}(x, y) \leq \mathrm{d}(F(x), F(y))$, whence $F \in \mathrm{Is}\,(\Gamma)$.

The converse however is not always true. At least an isometry F is injective on the set of vertices in general, since $F(x) = F(y)$ if and only if $\mathrm{d}(F(x), F(y)) = 0$ if and only if $\mathrm{d}(x, y) = 0$ if and only if $x = y$ by the metric properties of d. F does not need to be injective on the edges, since for two different geometric edges with the same borders, an isometry may map both edges to a single one, preserving injectivity for vertices (cf. Figure 1.11). If Γ is combinatorial, then each isometry $h \in \mathrm{Is}\,(\Gamma)$ is an injective endomorphism. Suppose $h(a) = h(b)$ for two edges a, b. Then $h\mathrm{o}(a) = \mathrm{o}(ha) = \mathrm{o}(hb) = h\mathrm{o}(b)$ gives $\mathrm{o}(a) = \mathrm{o}(b)$ and analogously $\mathrm{t}(a) = \mathrm{t}(b)$. This shows $a = b$ because Γ was supposed to be combinatorial. We verified:

Figure 1.11: A non-injective isometry

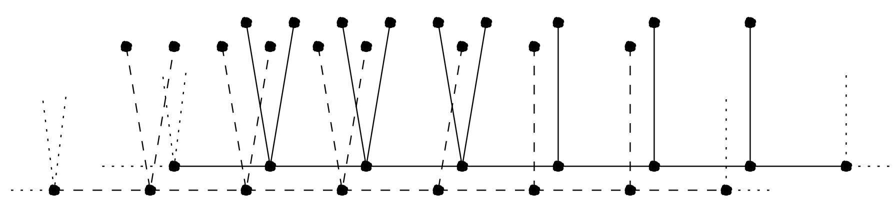


1.30 Lemma. *Every isometry of a combinatorial graph is an injective endomorphism.*

Clearly $\mathrm{Is}(\Gamma) = \mathrm{Aut}(\Gamma)$, if we consider only finite combinatorial graphs Γ. What can be said about surjectivity more generally? Surjectivity follows inductively from connection and local surjectivity (cf. Section 1.4.3). In turn local surjectivity holds for isometries on k-regular combinatorial graphs because of local injectivity (if k is finite), examples are k-regular trees $\mathcal{T}_k$.

Yet we need to work with more general locally finite trees in Chapter 3 and later. Unfortunately (see Figure 1.12) there are examples for isometries on locally finite trees, which are not surjective.

Figure 1.12: This locally finite tree (solid lines) has an isometry which is not surjective. A left shift may be applied. The shifted tree is drawn by dashed lines.



Fortunately we can give a proof of local surjectivity for isometries on combinatorial graphs which have a strong regularity. For $n, N \in \mathbb{N}_0$ and vertices $x \in \mathrm{V}\mathcal{T}$ we define $W_n(x) := \{y \in \mathrm{V}\mathcal{T} : \mathrm{d}(x,y) = n\}$ and $B_N(x) := \{y \in \mathrm{V}\mathcal{T} : \mathrm{d}(x,y) \leq N\} = \bigcup_{0 \leq n \leq N} W_n(x)$.

1.31 Proposition. *Suppose A is a locally finite combinatorial graph, $G < \mathrm{Aut}(A)$ and the quotient graph $B = G\backslash A$ is finite and connected. Then all isometries of A are locally surjective.*

Proof. If an isometry h is not locally surjective, then there is $y \in \mathrm{V}A$ with $|\mathrm{St}^A(y)| < |\mathrm{St}^A(hy)|$ since an isometry on a combinatorial graph is injective. We write $\mathbf{y} = \pi y$ and $N := \max\{\mathrm{d}(\mathbf{y}, \mathbf{x}) : \mathbf{x} \in \mathrm{V}B\} + 1$. Since A is combinatorial we can identify $\mathrm{St}^A(x) = \{e \in \mathrm{E}A : \mathrm{o}(e) = x\}$ with $\{x' \in \mathrm{V}A : \mathrm{d}(x, x') = 1\}$. We can prove by induction

For every $k \geq \deg(y)$ there is a vertex $y(k) \in \mathrm{V}A$, such that $\mathrm{St}^A(y) \subset B_N(y(k))$ and $|B_N(y(k))| \geq k$.

To prove the root of induction we put for $k = \deg(y)$ $y(k) = y$. Then $\mathrm{St}^A(y) \subset B_N(y(k))$, hence in particular $|B_N(y(k))| \geq k$.

For an induction step let $\mathbf{z} := \pi(hy(k))$ (π is the quotient map). For $n = \mathrm{d}(\mathbf{y}, \mathbf{z}) \leq N - 1$ there is a path p of length $n \geq 0$ in B from $\mathbf{y}$ to $\mathbf{z}$. By Lemma 1.29 we can choose a path q with origin y such that $p = \pi \circ q$ and we set

$$y(k+1) := \mathrm{t}(q).$$

For $y' \in \mathrm{St}^A(y)$ we get

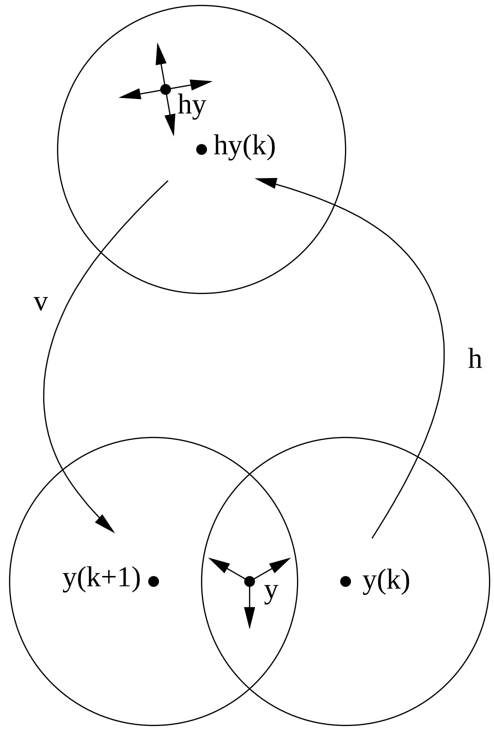
$$\begin{aligned} \mathrm{d}(y(k+1), y') \quad &\leq \quad \mathrm{d}(y(k+1), y) + \mathrm{d}(y, y') \quad = \quad \mathrm{d}(\mathrm{t}(q), \mathrm{o}(q)) + 1 \\ &\leq \quad \mathrm{len}(q) + 1 \quad = \quad n + 1 \quad \leq \quad N \end{aligned}$$

and hence $\mathrm{St}^A(y) \subset B_N(y(k+1))$.

Note $\pi(y(k+1)) = \pi(\mathrm{t}(q)) = \mathrm{t}(\pi(q)) = \mathrm{t}(p) = \mathbf{z}$. Therefore there is an automorphism $v \in G$ satisfying $y(k+1) = vh(y(k))$ (cf. Figure 1.13), in particular $v : B_N(hy(k)) \to B_N(y(k+1))$ is a bijection. Thus $|B_N(y(k))| < |B_N(hy(k))| = |B_N(y(k+1))|$, the inequality because of $\mathrm{St}(y) \subset B_N(y(k))$ and since h is an (injective) isometry. We get $|B_N(y(k+1))| \geq |B_N(y(k))| + 1 \geq k + 1$.

By Lemma 1.32 there is a constant B such that $|B_N(x)| \leq B$ independently of the vertex $x \in \mathrm{V}A$. For $k = \max\{|\mathrm{St}^A(y)|, B + 1\}$ we get the contradiction $|B_N(y(k))| \leq B < k \leq |B_N(y(k))|$. h is locally surjective. □

Figure 1.13: Induction pace



1.32 Lemma. *Suppose a group $G < \mathrm{Aut}(A)$ acts on a locally finite graph A with finite quotient $G\backslash A$, then $D_{\max} := \sup\{\deg(y) : y \in \mathrm{V}A\} < \infty$ and $|B_N(x)| \leq 1 + D_{\max} \sum_{n=0}^{N-1} (D_{\max} - 1)^n$ for all vertices $x \in A$.*

Proof. $\sup\{\deg(y) : y \in \mathrm{V}A\} = \sup\{\deg(y') : y' \in Gy, Gy \in V(G\backslash A)\} < \infty$, since all vertices in an orbit Gy have the same finite degree and since $G\backslash A$ is finite. The statement is proved by Lemma 1.33 □

1.33 Lemma. *If A is a graph and $D \in \mathbb{N}$ a constant, such that $\deg(x) \leq D$ for all vertices $x \in \mathrm{V}A$, then*

$$|B_N(x)| \leq 1 + D \sum_{n=0}^{N-1} (D-1)^n.$$

Proof. We put $R_n(x)$ for the set of reduced rays with origin x of length n for $n \in \mathbb{N}_0$ and $x \in \mathrm{V}A$. We are going to prove by induction, that $|R_n(x)| \leq D(D-1)^{n-1}$ for all $n \geq 1$. $R_1(x) = \mathrm{St}(x)$, hence $|R_1(x)| = |\mathrm{St}(x)| \leq D$. For all $n \geq 1$

$$\begin{aligned}
& R_{n+1}(x) = \\
= \quad & \{e_1, \ldots, e_{n+1} \text{ reduced path, } \mathrm{o}(e_1) = x\} \\
= \quad & \{e_1, \ldots, e_n, e : e_1, \ldots, e_n \text{ reduced path, } \mathrm{o}(e_1) = x, e \in \mathrm{St}(\mathrm{t}(e_n)) \setminus \overline{e_n}\} \\
= \quad & \bigcup_{e_1, \ldots, e_n \in R_n(x)} \{e_1, \ldots, e_n, e : e \in \mathrm{St}(\mathrm{t}(e_n)) \setminus \overline{e_n}\},
\end{aligned}$$

therefore $|R_{n+1}(x)| = \sum_{p \in R_n(x)} (|\mathrm{St}(\mathrm{t}(p))| - 1) \leq |R_n(x)|(D-1) \leq D(D-1)^n$. The map assigning the terminus to a path, $t : R_n(x) \to \mathrm{V}A$, has the property $W_n(x) \subset \mathrm{t}(R_n(x))$, since $\mathrm{d}(x, y) = n$ implies the existence of a path p from x to y of length n. This path is reduced (cf. the paragraph before equation (1.11)). Hence $|W_n(x)| \leq |R_n(x)|$ for all $n \geq 1$. Trivially $W_0(x) = 1$, so $|B_N(x)| = \sum_{n=0}^{N} |W_n(x)| \leq \sum_{n=0}^{N} |R_n(x)|$ finishes the proof. □

1.34 Proposition. *Suppose a group $G < \mathrm{Aut}\,(A)$ acts on a locally finite and connected combinatorial graph A with finite quotient graph $G \backslash A$, then* $\mathrm{Is}\,(A) = \mathrm{Aut}\,(A)$.

Proof. $\mathrm{Aut}\,(A) \subset \mathrm{Is}\,(A)$ has been discussed at the beginning of this section. Conversely, we saw also, that each isometry is injective, if A is combinatorial. To show surjectivity of a given isometry h, we can use Proposition 1.31 because connection of A implies connection of $G \backslash A$ by Lemma 1.14 and by surjectivity of the quotient morphism. Since by this proposition h is locally surjective and since A is connected, we can apply argument a) from Section 1.4.3 to gain surjectivity of h. This shows $\mathrm{Is}\,(A) \subset \mathrm{Aut}\,(A)$. □

1.35 Corollary. *Assume a group $G < \mathrm{Aut}\,(\mathcal{T})$ acts on a locally finite tree $\mathcal{T}$ with finite quotient graph $G \backslash \mathcal{T}$, then* $\mathrm{Is}\,(\mathcal{T}) = \mathrm{Aut}\,(\mathcal{T})$.

Proof. A tree is a connected combinatorial graph, hence the assertion follows from Proposition 1.34. □

Chapter 2

Edge-indexed graphs

In view of Part III, this Chapter can be given some motivation. Finite edge index graphs and geodesics constitute the basis whereon a dynamical system will be established there. These concepts are presented in the first section. Uni-modularity will have some importance in connection with isometry groups of the universal cover (cf. Chapter 3). The oriented line graph of an edge-indexed graph is introduced in Section 2.2. The connection properties of this line graph determine the ergodic properties of the dynamical system. Therefore in Sections 2.4 and 2.5 a classification of these graphs is presented.

2.1 Geodesics

To a finite connected graph A we add a new structure, a function

$$\mathrm{i}_A : \mathrm{E}A \longrightarrow \mathbb{N}$$

and call $\mathrm{i}(e)$ the *index* of an edge e. (A, i_A) is then called an *edge-indexed graph*. In diagrams we must assign two numbers to each geometric edge. The index of an (oriented) edge will be written closed to its origin (see Example 2.5). An edge-indexed graph (A, i_A) with constant indexing $\mathrm{i}(e) = 1$ for all $e \in \mathrm{E}A$ is denoted by $(A, 1)$. We say that an indexing i_2 of a graph A is *greater* than an indexing i_1, if $\mathrm{i}_2(e) \geq \mathrm{i}_1(e)$ for all edges $e \in \mathrm{E}A$ and write this as $\mathrm{i}_2 \geq \mathrm{i}_1$.

Additionally to paths and reduced paths we define another "type" of paths.

A *geodesic* of length $n \geq 1$ is a path

$$e_1, \dots, e_n$$

satisfying $b = \overline{a} \Rightarrow \mathrm{i}(b) > 1$ for all segments a, b of length two. Note that every reduced path is a geodesic. But there are more geodesics than reduced paths, because backtracking is allowed into edges e, whenever $\mathrm{i}(e) > 1$. Observe also, that $^{-}$ maps geodesics to geodesics, since $(b = \overline{a} \Rightarrow \mathrm{i}(b) > 1) \Leftrightarrow (b = \overline{a} \Rightarrow \mathrm{i}(\overline{a}) > 1) \Leftrightarrow (\overline{a} = \overline{\overline{b}} \Rightarrow \mathrm{i}(\overline{a}) > 1)$ for all segments a, b of length two.

An *infinite geodesic* is an infinite path where every segment of length two is a geodesic. We define

$$\mathcal{G}(A, \mathrm{i}_A)) := \{\text{bi-infinite geodesics in } (A, \mathrm{i}_A)\}$$

A path g is called a *closed geodesic* if and only if the composition gg is a geodesic.

2.2 Oriented line graphs

To an edge-indexed graph (A, i_A) we associate its *oriented line graph* $\mathcal{L}^+(A, \mathrm{i}_A)$. By Proposition 1.9 we can define an oriented graph by specifying a set of vertices and positive edges together with the borders of these positive edges. The set of vertices of $\mathcal{L}^+(A, \mathrm{i}_A)$ are the edges of A.

$$\mathrm{V}\mathcal{L}^+(A, \mathrm{i}_A) := \mathrm{E}A$$

The orientation of $\mathcal{L}^+(A, \mathrm{i}_A)$, is given by the set of positive edges

$$\mathrm{E}\mathcal{L}^+(A, \mathrm{i}_A)_+ := \{(a, b) \in \mathrm{E}A \times \mathrm{E}A : a, b \text{ is a geodesic in } (A, \mathrm{i}_A)\}.$$

Origin and terminus are $\mathrm{o}(a, b) = a$ and $\mathrm{t}(a, b) = b$. In diagrams we draw always the positive edge of each geometrical edge as an arrow.

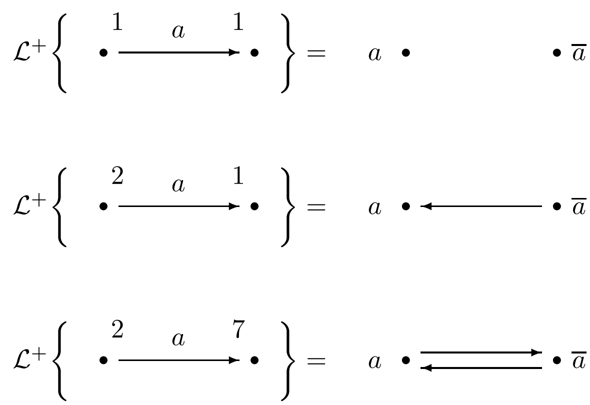
The oriented line graph is not a combinatorial graph in general, as the way of writing edges might suggest. If an edge-indexed graph has two geodesics a, b and b, a ($a \neq b$), then we obtain two positive edges (a, b) and (b, a). In particular $(b, a) \neq \overline{(a, b)}$.

The correspondence of Lemma 2.2 extends to infinite paths through application to their finite segments. We therefore obtain a one-to-one correspondence between the set $\mathcal{G}(A, \mathrm{i}_A)$ of bi-infinite geodesics of (A, i_A) and the set $\mathcal{P}\mathcal{L}^+(A, \mathrm{i}_A)$ of bi-infinite positive paths of $\mathcal{L}^+(A, \mathrm{i}_A)$.

2.1 Example (Derivation of oriented line graphs).

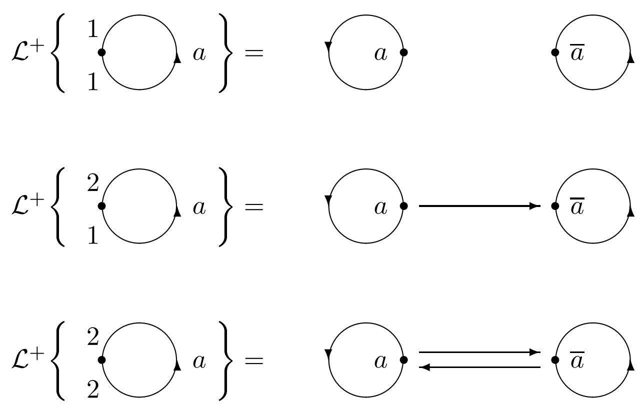
- from path_1 in Figure 2.1:

Figure 2.1: Oriented line graphs derived from path_1



- from circ_1 in Figure 2.2:

Figure 2.2: Oriented line graphs derived from circ_1



- As a generalization of the first graph in Figure 2.2 we have $\mathcal{L}^+(\text{circ}_n, 1) = \text{circ}_n \sqcup \text{circ}_n$. The oriented line graph of $(\text{circ}_n, 1)$ is the disjoint union of two copies of circ_n.

2.2 Lemma. *There is a one-to-one correspondence between geodesics of length greater equal than one in* (A, i_A) *and positive paths of length greater equal than zero in* $\mathcal{L}^+(A, i_A)$.

Proof. Geodesics of length one are exactly the edges, which correspond by definition to the vertices of $\mathcal{L}^+(A, \mathrm{i}_A)$ and represent there (positive) paths of length zero. We may map by α a geodesic $g = e_1, \ldots, e_n$ of length $n \geq 2$ to the sequence $(e_1, e_2), \ldots, (e_{n-1}, e_n)$ of positive edges of $\mathcal{L}^+(A, \mathrm{i}_A)$. The sequence is a edge sequence for a positive path of length $n - 1$.

Conversely we can map by β a positive path $p = (e_1, e_2), \ldots, (e_{n-1}, e_n)$ of length $n - 1 \geq 1$ to $e_1, \ldots, e_n$, which is a geodesic. The composition $\beta \circ \alpha$ respectively $\alpha \circ \beta$ is the identity on the set of geodesic segments of length greater zero in $((A, \mathrm{i}_A))$ respectively on the set of positive segments in $\mathcal{L}^+(A, \mathrm{i}_A)$. □

2.3 Lemma. *For every* $k > 0$ *an edge-indexed graph* (A, i_A) *has a closed geodesic of length* k *if and only if* $\mathcal{L}^+(A, i_A)$ *has a closed positive path of length* k.

Proof. By Lemma 2.2 we have a bijection between positive paths in $\mathcal{L}^+(A, \mathrm{i}_A)$ and geodesics of length greater zero in (A, i_A). If a positive path of length greater zero in $\mathcal{L}^+(A, \mathrm{i}_A)$ is closed, then its borders are equal. Thus the corresponding geodesic has the first edge equal to the last one. Removing one of them gives a closed geodesic in (A, i_A).

Conversely the edges sequence of a closed geodesic of length greater zero in (A, i_A) defines a vertex sequence of a closed positive path in $\mathcal{L}^+(A, \mathrm{i}_A)$ if we repeat the first edge at the end of the geodesic. The number of vertices of a path exceeds the number of defining edges by one concludes the proof. □

2.3 Unimodularity

The concept of unimodularity for edge-indexed graphs is in correspondence with unimodularity for the full group G_f introduced in Chapter 3 that will be used in Chapter 5. We define for edges $e \in \mathrm{E}\Gamma$

$$\Delta(e) := \frac{\mathrm{i}(\overline{e})}{\mathrm{i}(e)},$$

for paths of length $n \geq 1$ with edge sequence $e_1, \ldots, e_n$

$$\Delta(p) := \Delta(e_1) \cdots \Delta(e_n).$$

For convenience we put for paths p of length zero $\Delta(p) := 1$. An edge-indexed graph (A, i_A) is called *unimodular* if

$$\Delta(p) = 1 \tag{2.1}$$

holds for all closed paths p in (A, i_A). For two composable paths p and q there are easy to verify relations:

$$\begin{aligned} \Delta(pq) &= \Delta(p)\Delta(q) \\ \Delta(pq) &= \Delta(qp) \\ \Delta(\overline{p}) &= \tfrac{1}{\Delta(p)} \end{aligned} \tag{2.2}$$

2.4 Lemma. *An edge-indexed graph (A, i_A) is unimodular if and only if for all closed reduced paths p in A holds $\Delta(p) = 1$.*

Proof. If p is a closed path, then for any reversal $a, \overline{a}$ appearing as $p = p_1 a \overline{a} p_2$ respectively as $p = \overline{a} p_3 a$ one has

$$\Delta(p) = \Delta(p_1 p_2) \quad \text{respectively} \quad \Delta(p) = \Delta(p_3)$$

by equation (2.2). After successively removing such reversals of p, we obtain a closed reduced path q with $\Delta(p) = \Delta(q) = 1$ by assumption. The opposite direction is clear, since a closed reduced path is a closed path. □

2.5 Example (Unimodularity).

- The edge-indexed graph in Figure 2.3 forms a unimodular edge-indexed graph only for $k = l$.

Figure 2.3: circ_1 with indices $\{k, l\}$

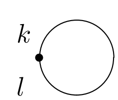
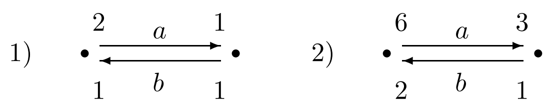


Figure 2.4: Non-unimodular and unimodular indexing for circ_2



- In Figure 2.4 there is circ_2 with a non-unimodular edge indexing 1) and a unimodular edge indexing 2).
- If Γ is a tree, then any edge indexing is unimodular. By Proposition 1.22 there are no closed reduced paths in Γ of positive lengths, hence Lemma 2.4 shows unimodularity.

2.4 Irreducible graphs

For an edge-indexed graph (A, i_A) we define for edges $a, b \in \mathrm{E}A$ the relation

$$a \sim b :\Leftrightarrow \text{ a geodesic } g \text{ with } \mathrm{len}(g) \geq 2 \text{ leads from } a \text{ to } b. \tag{2.3}$$

As a first observation, the concatenation $g \wedge h$ of two geodesics g and h is a geodesic, hence $\sim$ is a transitive relation. Note also, that $a \sim b \Leftrightarrow \overline{b} \sim \overline{a}$ since $\bar{\ }$ maps geodesics to geodesics.

By Lemma 2.2, a positive path of positive length joins a with b for two vertices $a, b \in \mathcal{L}^+(A, \mathrm{i}_A)$, if and only if a geodesic in (A, i_A) of length greater equal two leads from the edge a to the edge b. It is therefore, that each two vertices in $\mathcal{L}^+(A, \mathrm{i}_A)$ are joined by a positive path of length greater zero if and only if $a \sim b$ for all edges $a, b \in \mathrm{E}A$. In this case both (A, i_A) and $\mathcal{L}^+(A, \mathrm{i}_A)$ are called *irreducible*.

An edge d of an edge-indexed graph (A, i_A) is called a *dead end* if $\deg(\mathrm{o}(\overline{d})) = 1$ and $\mathrm{i}_A(\overline{d}) = 1$. This section gives a clear criterion for irreducibility of edge-indexed graphs, which are

finite, connected
and without dead ends. (2.4)

Demanding connection of the edge index graph is no restriction to the problem. A non-connected graph, that has two connected components with edges, can not be irreducible (cf. Lemma 1.15). It is natural also to exclude dead ends. A path starting with a dead end d must have as a second edge $\overline{d}$ but $d, \overline{d}$ is not a geodesic in this case. Additionally, in view of the dynamical system of Chapter 5, which is based on bi-infinite geodesics of (A, i_A), dead ends have no significance, since they simply would not appear on bi-infinite geodesics. It is of course a restriction to look at finite graphs only.

Since some arguments in Chapter 5 work only for unimodular edge-indexed graphs, at the end of the section we will have a particular view of edge indexed graphs as in (2.4), which are **unimodular**.

We write $C^+(N) = \{[0,1], \dots, [N-1,0]\}$ for the (standard) orientation of the oriented graph circ_N and define

$$\begin{aligned} \mathcal{BG} \quad := \quad & \{(\mathrm{circ}_N, \mathrm{i}) : \mathrm{i}(e) = 1 \text{ for all } e \in C^+(N), N \in \mathbb{N}\} \\ & \cup \quad \{(\mathrm{circ}_N, \mathrm{i}) : \mathrm{i}(e) = 1 \text{ for all } e \in \overline{C^+(N)}, N \in \mathbb{N}\}, \end{aligned}$$

as well as (*nice graphs*)

$$\mathcal{NG} = \{(A, \mathrm{i}_A) \text{ as in (2.4)} : a \sim \overline{a} \text{ for all edges } a \in \mathrm{E}A\}\,.$$

The classification of graphs as (2.4) will be done in two lines. First $\mathcal{NG}$ will be shown to be the compliment of $\mathcal{BG}$ up to Corollary 2.9. Then by independent arguments the graphs of $\mathcal{NG}$ will be presented as exactly the irreducible ones up to Theorem 1.

2.6 Lemma. $\mathcal{NG} \cap \mathcal{BG} = \emptyset$.

Proof. If $(A, \mathrm{i}_A) \in \mathcal{BG}$ then $A = \mathrm{circ}_N$ and we can assume that $\mathrm{i}_A([n, n+1]) = 1$ for all $n \in \mathbb{Z}_N$. The only geodesic with first edge $[0,1]$ of length $n+1 \geq 2$ is then the reduced path $[0,1], \dots, [n, n+1]$. In particular $[0,1] \not\sim \overline{[0,1]}$, and hence $(A, \mathrm{i}_A) \notin \mathcal{NG}$. □

2.7 Proposition. *Suppose* Γ *is a finite connected graph and* $e \in \mathrm{E}\Gamma$*. If for all* $n \geq 1$ *there is a unique reduced path* g_n *of length* n *with first edge* e*, then* $\Gamma = \mathrm{circ}_N$ *for some* $N \in \mathbb{N}$*.*

Proof. By uniqueness of the reduced paths g_n, there is a ray $e_1, e_2, e_3, \dots$ such that $g_n = (e_1, \dots, e_n)$ for $n \geq 1$. We define the subgraph $A = \langle\{\mathrm{t}(e_i) : i \in \mathbb{N}\}\rangle$ of Γ. Since A is finite and $\{e_i : i > 1\} \subset \mathrm{E}A$, one has $e_k = e_l$ for some $k < l$ in

$\mathbb{N}$, whence $(e_2, \dots, e_l)$ is not an injective reduced path. By Proposition 1.22 A is not a tree and must therefore contain a circuit of some length $N \in \mathbb{N}$. This gives an injection

$$\gamma : \mathrm{circ}_N \longrightarrow \Gamma.$$

We carry on showing, that γ is surjective. Since $\gamma(i) \in \mathrm{V}A$ for all $i \in \mathrm{V}\mathrm{circ}_N$, $\gamma(i) = \mathrm{t}(e_k)$ for some $k \in \mathbb{N}$. If $\deg(\gamma(i)) > 2$ then $\deg(\mathrm{t}(e_k)) > 2$, which is a contradiction to the assumed uniqueness. This shows, that γ is locally surjective hence γ is surjective (cf. Section 1.4.3). □

2.8 Proposition. *Suppose* (A, i_A) *is an edge-indexed graph as in* (2.4). *If* $a \not\sim \overline{a}$ *for an edge* $a \in \mathrm{E}A$*, then* $A = \mathrm{circ}_N$ *for some* $N \in \mathbb{N}$.

Proof. As there are no dead ends, for all $n \in \mathbb{N}$ there is a geodesic $(e_1, \dots, e_n)$ with first edge $e_1 = a$. Such a geodesic is a reduced path, since if $e_{i+1} = \overline{e_i}$ then $(e_1, \dots, e_i, \overline{e_i}, \dots, \overline{e_1})$ is a geodesic of length greater one, hence $a \sim \overline{a}$ in contradiction to assumptions. We want to show that for each $n \in \mathbb{N}$ there is only one such geodesic. Then Proposition 2.7 concludes the proof.

We show first that any geodesic $(e_1, \dots, e_n)$ with first edge a is a segment of a closed geodesic. Such a geodesic may be extended to a geodesic ray $(e_1, \dots, e_n, e_{n+1}, \dots)$. Since $\mathrm{E}A$ is finite, there are two natural numbers k, l with $n < k < l$ such that $e_k = e_l$. Note $\mathrm{t}(e_{l-1}) = \mathrm{o}(e_l) = \mathrm{o}(e_k) = \mathrm{t}(e_{k-1}) = \mathrm{o}(\overline{e_{k-1}})$. If $e_{k-1} \neq e_{l-1}$ then $e_{l-1}, \overline{e_{k-1}}$ is reduced, hence $(e_1, \dots, e_{l-1}, \overline{e_{k-1}}, \dots, \overline{e_1})$ is a geodesic of length greater one, thus $a \sim \overline{a}$ in contradiction to assumptions. This shows $e_{k-1} = e_{l-1}$. Inductively we get

$$e_{k-i} = e_{l-i} \quad \text{for all } 0 \leq i < k,$$

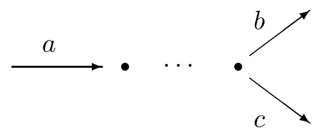
so $(e_1, \dots, e_l)$ is periodic with period $(l-k)$. The path $(e_1, \dots, e_n)$ as a segment of $(e_1, \dots, e_l)$ is periodic to the same period, hence is a segment of the closed geodesic

$$\underbrace{(e_1, \dots, e_{l-k}) \cdots (e_1, \dots, e_{l-k})}_{m \quad \text{times}}$$

for $m = \left[\frac{n}{l-k}\right] + 1$, where $[x]$ denotes the greatest integer i with $i < x$.

For uniqueness it is sufficient to show $c = b$ whenever two geodesics agb and agc are given by a path g with $\mathrm{len}(g) \geq 0$ and two edges b, c. Suppose this is

Figure 2.5: A junction in an edge-indexed graph



not the case (cf. Figure 2.5). As we saw above, both geodesics are segments of closed geodesics, so there are geodesics

$$\begin{aligned} &ag(b_0, b_1, \ldots, b_k) \quad \text{and} \\ &ag(c_0, c_1, \ldots, c_l) \end{aligned}$$

with $b_0 = b \neq c = c_0$ and $b_k = c_l = a$ $(1 \leq k, l)$. We can choose $m = \min\{i : b_i = c_j \text{ for some } 0 < j \leq l\}$ and $p \in \mathbb{N}$ with $b_m = c_p$. Obviously $(m, p > 0)$ $b_{m-1} \neq c_{p-1}$, hence $b_{m-1}, \overline{c_{p-1}}$ is reduced. Putting $B = (b_0, \ldots, b_{m-1})$ and $C = (c_0, \ldots, c_{p-1})$ the path $B\overline{C}$ is a geodesic with first edge b and last edge $\overline{c}$. This shows $b \sim \overline{c}$ and by transitivity $a \sim \overline{a}$ in contradiction to assumptions. □

2.9 Corollary. *The set of all edge-indexed graphs as in (2.4) is the disjoint union* $\mathcal{NG} \sqcup \mathcal{BG}$.

Proof. By Lemma 2.6 the union is disjoint. If an edge-indexed graph (A, i_A) is not an element of $\mathcal{NG}$, then by definition $a \not\sim \overline{a}$ for some edge. So by Proposition 2.8 $A = \mathrm{circ}_N$ for some $N \in \mathbb{N}$. We can assume $[0,1] \not\sim \overline{[0,1]}$ by a symmetry argument. Obviously $[n, n+1] \sim [m, m+1]$ and $\overline{[n, n+1]} \sim \overline{[m, m+1]}$ for all $m, n \in \mathbb{Z}_N$. If $\mathrm{i}_A\overline{[k, k+1]} > 1$ for any $k \in \mathbb{Z}_N$ then $[k, k+1]\overline{[k, k+1]}$ is a geodesic, thus $[k, k+1] \sim \overline{[k, k+1]}$. Since $[0,1] \sim [k, k+1]$ and $\overline{[k, k+1]} \sim \overline{[0,1]}$ follows $[0,1] \sim \overline{[0,1]}$ in contradiction. Thus $(A, \mathrm{i}_A) \in \mathcal{BG}$. □

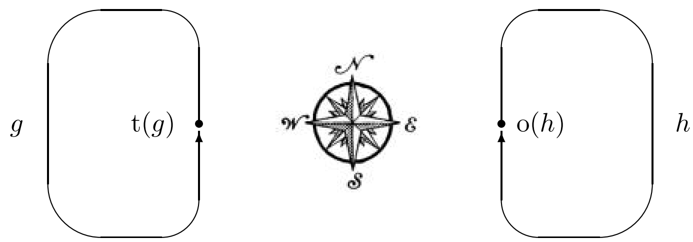
2.10 Lemma. *Every geodesic in an edge-indexed graph* $(A, \mathrm{i}_A) \in \mathcal{NG}$ *is a segment of a closed geodesic.*

Proof. If the geodesic has length zero, there is nothing to show. So let $g = (e_1, \ldots, e_n)$ be a geodesic of length $n \geq 1$. By definition of $\mathcal{NG}$ there are geodesics $h_1 = (e_n, \ldots, \overline{e_n})$ and $h_2 = (\overline{e_1}, \ldots, e_1)$ of lengths ≥ 2. We can define h_2' by $h_2' e_1 = h_2$. The geodesic $g \wedge h_1 \wedge \overline{g} \wedge h_2'$ is then closed. □

2.11 Proposition. *If g and h are two closed geodesics of an edge-indexed graph (A, i_A) with $\mathrm{t}(g) = \mathrm{o}(h)$, then one of the paths gh or $g\overline{h}$ is a closed geodesic.*

Proof. Since $\mathrm{t}(\overline{g}) = \mathrm{t}(g)$, $\mathrm{o}(h) = \mathrm{o}(\overline{h})$ and since the inverse of a closed geodesic is a closed geodesic, the assumptions of this Proposition are invariant under a reflection of the diagram about the West-East axis (cf. Figure 2.6). Nor is the

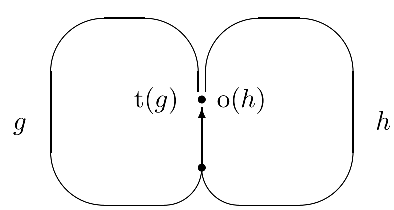
Figure 2.6: Composable closed geodesics



statement of the Proposition changed by this transformation, because $\overline{g}\overline{h}$ is a closed geodesic if and only if gh is a closed geodesic and $\overline{g}h$ is a closed geodesic if and only if $g\overline{h}$ is a closed geodesic.

We can assume, that $\mathrm{len}(g), \mathrm{len}(h) \geq 1$ and write $g = (a_1, \ldots, a_n)$ and $h = (b_1, \ldots, b_m)$, excluding trivial cases. If $g\overline{h}$ is not a closed geodesic, then $(a_n, \overline{b_m})$ or $(\overline{b_1}, a_1)$ is not a geodesic, whence $\overline{b_m} = \overline{a_n}$ or $a_1 = \overline{\overline{b_1}}$ or equivalently $a_n = b_m$ or $\overline{a_1} = \overline{b_1}$. Performing a reflection about the W-E axis, if necessary, we can thus assume $a_n = b_m$ (cf. Figure 2.7). Now, since hh is a geodesic gh

Figure 2.7: Composition of closed geodesics



is one, since gg is a geodesic hg is one and we can conclude, that gh is a closed geodesic. □

2.12 Proposition. *For* $(A, \mathrm{i}_A) \in \mathcal{NG}$ *the relation* (2.3) *is an equivalence relation. It has one or two equivalence classes. In the latter case the classes are orientations for* A.

Proof. Transitivity follows directly from definition of $\sim$. By definition of $\mathcal{NG}$ one has $e \sim \overline{e}$ for all edges e. Hence for any edge a one verifies reflexivity by $a \sim \overline{a} \sim \overline{\overline{a}} = a$ trough transitivity. For symmetry we assume $a \sim b$, i.e. there is a geodesic agb. By Lemma 2.10 this geodesic can be extended to a closed geodesic $agbh$. Then $agbhagbh$ is a geodesic, thus also the segment bha.

We prove that $\sim$ has at most two classes if we show, that each class has at least one edge from every geometric edge. Therefore it is sufficient to verify for each pair of edges a, b, that $a \sim b$ or $a \sim \overline{b}$. By Lemma 2.10, a can be extended to a closed geodesic ga, and b can be extended to a closed geodesic bh. Be connection of A, is a reduced path p joining $\mathrm{t}(a)$ with $\mathrm{o}(b)$. By the same argument as above we can extend p to a closed geodesic pq. With Proposition 2.11 $gapq$ or $ga\overline{q}\,\overline{p}$ is a closed geodesic. Reordering theses two closed paths cyclicly, one of the paths

$$\{\, q\,g\,a\,p, \quad \overline{p}\,g\,a\,\overline{q} \,\}$$

will still be a closed geodesic, now with terminus $\mathrm{o}(b)$. A second application of the same proposition shows that there is at least one geodesic among

$$\big\{\, q\,g\,a\,p\,b\,h, \quad q\,g\,a\,p\,\overline{h}\,\overline{b}, \quad \overline{p}\,g\,a\,\overline{q}\,b\,h, \quad \overline{p}\,g\,a\,\overline{q}\,\overline{h}\,\overline{b} \,\big\}\,.$$

We can conclude $a \sim b$ or $a \sim \overline{b}$ for all edges a, b.

If one class is not an orientation, then it includes an edge e together with its inverse edge $\overline{e} \sim e$, since the same class intersects each geometrical edge in one edge by the argument of the above paragraph. Further, the property $a \sim e$ or $a \sim \overline{e}$ for all edges a shows that there is only one class. One can deduce, that in case of two classes, the classes are orientations of A. □

Theorem 1. *An edge-indexed graph* (A, i_A) *as in* (2.4) *is irreducible if and only if* $(A, \mathrm{i}_A) \in \mathcal{NG}$.

Proof. If $(A, \mathrm{i}_A) \in \mathcal{NG}$ then $a \sim \overline{a}$ for all edges. By Proposition 2.12 $\sim$ has only one class, i.e. (A, i_A) is irreducible. Conversely, if (A, i_A) is irreducible, then in particular $a \sim \overline{a}$ for all edges, thus $(A, \mathrm{i}_A) \in \mathcal{NG}$. □

2.13 Corollary. *A unimodular edge-indexed graph as in* (2.4) *is irreducible if and only if* $(A, \mathrm{i}_A) \neq (\mathrm{circ}_N, 1)$ *for all* $N \in \mathbb{N}$.

Proof. If (A, i_A) is irreducible, then by Theorem 1 $(A, \mathrm{i}_A) \in \mathcal{NG}$. As $(\mathrm{circ}_N, 1) \in \mathcal{BG}$, Corollary 2.9 shows $(A, \mathrm{i}_A) \neq (\mathrm{circ}_N, 1)$.

If (A, i_A) is not irreducible, then by these same Theorem and Corollary $(A, \mathrm{i}_A) \in \mathcal{BG}$. So $A = \mathrm{circ}_N$ for some $N \in \mathbb{N}$ and we assume $\mathrm{i}_A[n, n+1] = 1$ for all $n \in \mathbb{Z}_N$. Now by unimodularity one has

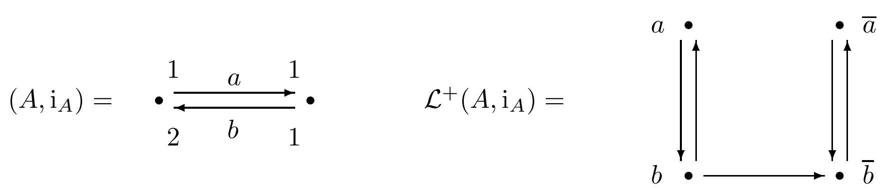
$$\begin{aligned} \mathrm{i}_A\overline{[k,k+1]} &\leq \prod_{n=0}^{N-1} \mathrm{i}_A\overline{[n,n+1]} = \prod_{n=0}^{N-1} \frac{\mathrm{i}_A\overline{[n,n+1]}}{\mathrm{i}_A[n,n+1]} \\ &= \Delta([0,1], \ldots, [N-1, 0]) = 1. \end{aligned}$$

Since $\mathrm{i}_A(e) \geq 1$ for all edges, one has $\mathrm{i}_A(e) = 1$ for all edges, thus $(A, \mathrm{i}_A) = (\mathrm{circ}_N, 1)$. ☐

2.14 Example (Irreducibility).

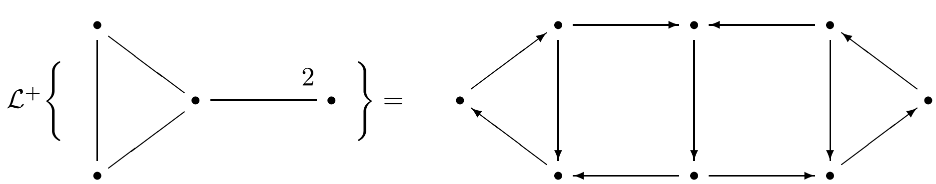
- An example for $(A, \mathrm{i}_A) \in \mathcal{BG}$. There is no positive path from $\overline{b}$ to b (cf. Figure 2.8).

Figure 2.8: A non-irreducible graph



- Figure 2.9 shows an example of an irreducible edge indexed graph.

Figure 2.9: An irreducible graph



- Example 2.1 is also interesting in the light of irreducibility.

2.5 Transitive graphs

Given an edge-indexed graph (A, i_A) as in (2.4) we say that both (A, i_A) and $\mathcal{L}^+(A, \mathrm{i}_A)$ are *transitive* if and only if there is a natural number $N \in \mathbb{N}$ such that for all $n \geq N$ and all vertices $x, y \in \mathrm{V}\mathcal{L}^+(A, \mathrm{i}_A)$ a positive path of length n joins x with y. Of course this statement can by translated by Lemma 2.2 to geodesics in (A, i_A).

A transitive graph is trivially irreducible, nevertheless we are going to give criterions for transitivity independently of $\mathcal{NG}$ for all edge-indexed graphs as in (2.4), to have a clearer picture. In praxis one may first check, whether a graph is irreducible or not.

For the following Lemma and Proposition only we use the term "transitive" in the obvious way for more general oriented graphs as the ones we introduced. For two integer numbers $p, d \neq 0$ one says d *divides* p if and only if there is an integer r with $p = dr$. An integer $d \neq 0$ is called a *common divisor* of two integers $p, q \neq 0$ if and only if d divides both p and q. Two integers $p, q \neq 0$ are called *coprime* if and only if 1 and -1 are the only common divisors of p and q.

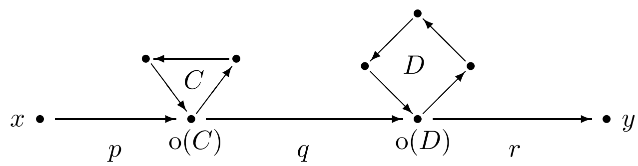
2.15 Lemma. *A transitive oriented graph has two closed positive paths of coprime lengths.*

Proof. We may fix a vertex x. Since the graph is transitive, there are positive paths from x to x for every length greater equal than some number $N \in \mathbb{N}$. So there is closed positive path of length N and one of length $N+1$. If N and $N+1$ were not coprime, then $N = kd$ and $N+1 = ld$ for $|d| \geq 2$ and two integers $k \neq l$, hence $2 \leq |l-k||d| = |N+1-N| = 1$. □

2.16 Proposition. *A finite and positively connected oriented graph G which has two closed positive paths of coprime lengths is transitive.*

Proof. Suppose C and D are two closed positive paths of length $c \geq 1$ and $d \geq 1$ respectively. Since G is positively connected, for each pair $x, y \in \mathrm{V}G$ there is a positive path p from x to $\mathrm{o}(C)$, a positive path q from $\mathrm{t}(C)$ to $\mathrm{o}(D)$ and a positive path r from $\mathrm{t}(D)$ to y. We set $\mathrm{len}(x, y) := \mathrm{len}(p) + \mathrm{len}(q) + \mathrm{len}(r)$. The path $pCqDr$ (Figure 2.10) is a positive path as a composition of positive paths. C and D are closed thus also $pC^s qD^t r$ is a positive path for all s and t greater

Figure 2.10: Circuitous routes about closed positive paths



equal zero ($C^0 := \mathrm{o}(C) = \mathrm{t}(C)$, $C^1 := C$, $C^2 := CC$ the composition of C with C, and so on).

If c and d are coprime, there are two integers $\eta, \xi \in \mathbb{Z}$ satisfying

$$1 = \eta c + \xi d$$

(cf. [15], Theorems 23 and 25, for existence — η and ξ may be calculated directly by the Euclidean Algorithm).

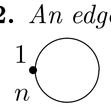
One of the numbers η or ξ is not positive, say $-\eta \geq 0$. We set $m = \min(c, d)$ and find the following positive paths P_k $(1 \leq k \leq m)$:

$$\begin{aligned}
P_1 &= pC^{(m-1)(-\eta)}qr \\
P_2 &= pC^{(m-2)(-\eta)}qD^{\xi}r \\
&\vdots \\
P_k &= pC^{(m-k)(-\eta)}qD^{(k-1)\xi}r \\
&\vdots \\
P_{m-1} &= pC^{(-\eta)}qD^{(m-2)\xi}r \\
P_m &= pqD^{(m-1)\xi}r
\end{aligned}$$

They all join x with y. The length of P_k is

$$\begin{aligned}
\mathrm{len}(P_k) &= \mathrm{len}(x, y) + c(m-k)(-\eta) + d(k-1)\xi \\
&= \mathrm{len}(x, y) + c(m-1)(-\eta) + (k-1).
\end{aligned}$$

Each of these positive paths can be augmented by inserting an additional closed positive path of length m. Hence for all numbers greater equal than $L(x, y) := \mathrm{len}(x, y) + c(m-1)(-\eta)$ there is a positive path from x to y. Since G has finitely many vertices, $L = \max_{x,y \in \mathrm{V}G} L(x, y)$ exists, thus G is transitive. □

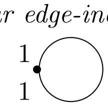
Theorem 2. *An edge-indexed graph (A, i_A) as in (2.4) is transitive if and only if $(A, \mathrm{i}_A) \neq$* 

for all $n \in \mathbb{N}$ and there are two closed geodesics of coprime lengths.

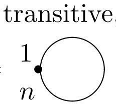
Proof. The inequality for (A, i_A) means $A \neq \mathrm{circ}_1$ or $\mathrm{i}_A \geq 2$. Assuming this case and the existence of coprime geodesics, we show in the next paragraph, that $(A, \mathrm{i}_A) \in \mathcal{NG}$. Theorem 1 shows then, that $\mathcal{L}^+(A, \mathrm{i}_A)$ is positively connected and we can argue with Proposition 2.16 that (A, i_A) is transitive.

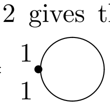
One may assume $(A, \mathrm{i}_A) \notin \mathcal{NG}$. Then Corollary 2.9 tells us that $A = \mathrm{circ}_N$ for some $N \geq 1$. If $N = 1$ then $\mathrm{i}_A \geq 2$ by assumption, hence $(A, \mathrm{i}_A) \in \mathcal{NG}$. Else $(N \geq 2)$ one may put $\mathrm{i}\overline{[i, i+1]} = 1$ for all $i \in \mathbb{Z}_N$. A geodesic g with first edge $[k, k+1]$ has then only edges of the same (standard) orientation. In particular, if g is closed, then the last edge of g is $[k-1, k]$, g has length $\mathrm{len}(g) = 0 \mod N$ and every closed geodesic with last edge $[k-1, k]$ has $[k, k+1]$ as fist edge. Now if a closed geodesic g has as first edge $\overline{[m, m+1]}$, then the last edge of the closed geodesic $\overline{g}$ equals $[m, m+1]$, whence $\mathrm{len}(g) = \mathrm{len}(\overline{g}) = 0 \mod N$, too. This shows that all closed geodesics are of length $0 \mod N$, which is a contradiction to the existence of a pair of geodesics with coprime lengths, since we assumed $N \geq 2$.

For the opposite direction, $A = \mathrm{circ}_1$ and $\mathrm{i}_A[0, 1] = 1$, one has $(A, \mathrm{i}_A) \in \mathcal{BG}$, hence by Corollary 2.9 $(A, \mathrm{i}_A) \notin \mathcal{NG}$ thus by Theorem 1 (A, i_A) is not irreducible, hence is not transitive. Likewise if there is no pair g, h of closed geodesics with coprime lengths then by Lemma 2.3 there is no pair p, q of closed positive paths in $\mathcal{L}^+(A, \mathrm{i}_A)$ with coprime lengths. Lemma 2.15 shows that $\mathcal{L}^+(A, \mathrm{i}_A)$ is not transitive. □

2.17 Corollary. *A unimodular edge-indexed graph* (A, i_A) *as in* (2.4) *is transitive if and only if* $(A, \mathrm{i}_A) \neq$ 

and there are two closed geodesics of coprime lengths.

Proof. If (A, i_A) is transitive, then Theorem 2 gives the result immediately. Contrarily $(A, i_A) =$ 

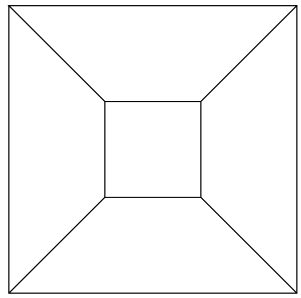
implies $(A, i_A) =$ 

by unimodularity. □

2.6 Examples

- The graph formed by the geometric edges of a cube can be drawn as the diagram in Figure 2.11. With indexing constant to one this edge-indexed graph is irreducible by Corollary 2.9 and Theorem 1 (it has a vertex of

Figure 2.11: The cube

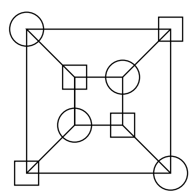


degree three unlike a graph circ_N). It is irreducible for any indexing i ≥ 1, too, i.e. for all indexings. Is it transitive?

A graph is called *bipartite*, if there is a partition of the vertex set into two subsets such that every edge has one border in each of the sets. Equivalently, a graph is bi-partite if and only if it has no closed path of odd length. This is shown in [14] for combinatorial graphs and is also true for graphs in general (Corollary A.1).

A bipartite graph can obviously not be transitive. We obtain such a partition for the cube into a "square"-subset and a "circle"-subset of its vertices (Figure 2.12). The cube is bipartite and hence not transitive.

Figure 2.12: Bipartition of the cube



- The graph formed by the geometric edges of a tetrahedron is drawn in Figure 2.13. With indexing constant to one it is irreducible and has both a closed geodesic of length three and a closed geodesic of length four. It is transitive.

- The edge-indexed graph in Figure 2.14 is irreducible and transitive.

Figure 2.13: The tetrahedron

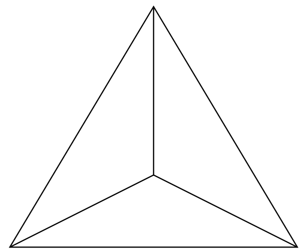
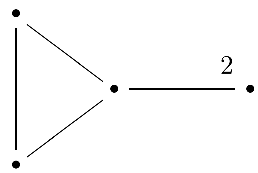
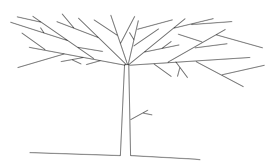


Figure 2.14: Another transitive graph



- A finite tree (see Figure 2.15 for the diagram of a finite tree) without dead ends is irreducible under any edge indexing. We can argue with Corollary 2.9 and Theorem 1 since trees have no circuits. In the first

Figure 2.15: A finite tree



example on page 40 we saw, that bipartite graphs are not transitive. We can construct a bipartition for any tree $\mathcal{T}$. We fix a vertex x and define

$$\begin{aligned} X_1 &= \{y \in \mathrm{V}\mathcal{T} : \mathrm{d}(x,y) \text{ is even } \} \\ X_2 &= \{y \in \mathrm{V}\mathcal{T} : \mathrm{d}(x,y) \text{ is odd } \}. \end{aligned}$$

The inequality

$$|\mathrm{d}(x, \mathrm{o}(e)) - \mathrm{d}(x, \mathrm{t}(e))| \leq \mathrm{d}(\mathrm{o}(e), \mathrm{t}(e)) \tag{2.5}$$

follows directly from the triangle inequality of the metric d. The left hand side of (2.5) is positive by Lemma 1.26. If for an edge e both $\mathrm{o}(e)$ and $\mathrm{t}(e)$

are in the same set X_1 or in X_2 then the difference in (2.5) is even and the contradiction $2 \leq \mathrm{d}(\mathrm{o}(e), \mathrm{t}(e)) = 1$ shows therefore, that X_1, X_2 is a bipartition for $\mathcal{T}$. A tree is not transitive under any edge indexing.

Part II

Trees

Chapter 3

Covering trees for graphs

The construction of a fundamental group and a universal cover to a finite connected edge-indexed graph will be organised in the first two sections. The universal cover is a tree. The fundamental group defines a left action on the universal cover and provides a quotient morphism from the universal cover back to the graph (cf. Figure 3.1). This morphism will be written in Section 3.3. Its local behavior will be discussed there as well. The construction involves *free*

Figure 3.1: Construction of the universal cover

edge-indexed graph (A, i_A)

$\downarrow$

graph of groups $\mathbb{A} = (A, \mathcal{A})$

$\downarrow$

fundamental group $G = \pi_1(\mathbb{A}, x_0)$

$\downarrow$

universal cover $\mathcal{T} = \widetilde{(\mathbb{A}, x_0)}$

$\downarrow$

quotient $A = G \backslash \mathcal{T}$

groups and *presentations of groups*. For an introduction into these topics, Chapter 11 of [16] may be a good source. Bass provides in [7] all central proofs except for Theorem 3, where the author refers to Serre [6]. The remaining space of the present chapter covers properties of the isometry group of universal covers, in particular its topology, and ends with some examples.

3.1 The fundamental group

The fundamental group is being constructed relative to a graph of groups and can afterwards be related to an edge-indexed graph.

3.1 Definition. A *graph of groups* $\mathbb{A} = (A, \mathcal{A})$ consists of a finite, connected graph A, groups $\mathcal{A}_x$ for all vertices $x \in \mathrm{V}A$, groups $\mathcal{A}_e = \mathcal{A}_{\overline{e}}$ for all edges $e \in \mathrm{E}A$ and monomorphisms (i.e. injective homomorphisms)

$$\alpha_e : \mathcal{A}_e \longrightarrow \mathcal{A}_{\mathrm{o}(e)}$$

for all edges $e \in \mathrm{E}A$. We abbreviate for $x = \mathrm{o}(e)$

$$\mathcal{A}_{x/e} := \mathcal{A}_x / \alpha_e \mathcal{A}_e \quad \text{resp.} \quad \mathrm{i}(e) := |\mathcal{A}_{x/e}| = [\mathcal{A}_x : \alpha_e \mathcal{A}_e]$$

the set family of left cosets respectively the number of left cosets of $\alpha_e \mathcal{A}_e$ in $\mathcal{A}_x$.

We want to associate a graph of groups to a given edge-indexed graph (A, i_A) such that the indexing i_A coincides with the numbers i of the graph of groups. This can be achieved by setting all groups equal to $\mathbb{Z}$ and choosing as monomorphisms

$$\alpha_e : n \mapsto \mathrm{i}(e)n$$

for edges $e \in \mathrm{E}A$. It may be interesting, that in the case of a finite graph A, which is assumed here, Corollary 2.5 in [11] allows us to choose all groups as finite groups if and only if (A, i_A) is uni-modular. However we will not make use of this result.

The *path group* of a graph of groups $\mathbb{A} = (A, \mathcal{A})$ is defined as

$$\pi(\mathbb{A}) = [(*_{x \in \mathrm{V}A} \mathcal{A}_x) * \mathrm{F}(\mathrm{E}A)]/\mathrm{R},$$

where F denotes the free group generated by $\mathrm{E}A$, $*$ denotes the free product of groups and R is the smallest normal subgroup imposing the relations

$$\begin{aligned} \overline{e} &= e^{-1} \qquad \text{and} \\ e\alpha_{\overline{e}}(s)e^{-1} &= \alpha_e(s) \end{aligned} \tag{3.1}$$

for all $e \in \mathrm{E}A$ and all $s \in \mathcal{A}_e = \mathcal{A}_{\overline{e}}$.

3.2 Note. In the special case $\mathcal{A}_e = \mathcal{A}_x = 1$ for all $e \in \mathrm{E}A$ and $x \in \mathrm{V}A$ we obtain simply $\pi(\mathbb{A}) = \mathrm{F}(\mathrm{E}A)/\mathrm{R}$ with relations $\overline{e} = e^{-1}$ for all $e \in \mathrm{E}A$.

A *π-path* of *length* $n \geq 0$ in $(A, \mathcal{A})$ is a sequence

$$\gamma = (g_0, e_1, g_1, e_2, \ldots, g_{n-1}, e_n, g_n),$$

where $e_1, \ldots, e_n$ is a path in A with vertex sequence $x_0, \ldots, x_n$ and $g_i \in \mathcal{A}_{x_i}$ (a π-path of length zero is an element $g_0 \in \mathcal{A}_{x_0}$). γ is then called a π-path *from* x_0 *to* x_n. We set

$$|\gamma| = g_0 e_1 g_1 \cdot \ldots \cdot g_{n-1} e_n g_n \in \pi(\mathbb{A})$$

and define for all $x, y \in \mathrm{V}A$

$$\pi[x, y] := \{|\gamma| \in \pi(\mathbb{A}) : \gamma \text{ is a } \pi\text{-path from } x \text{ to } y\}.$$

A π-path $\gamma = (g_0, e_1, g_1, e_2, \ldots, g_{n-1}, e_n, g_n)$ is called *reduced*, if and only if either $n = 0$ and $g_0 \neq 1$ or $n \geq 1$ and $e_{i+1} = \overline{e_i} \Rightarrow g_i \notin \alpha_{\overline{e_i}} \mathcal{A}_{e_i}$.

Theorem 3 (J.P.Serre [6]). *If γ is a reduced π-path in $\mathbb{A}$, then $|\gamma| \neq 1$ in $\pi(\mathbb{A})$.*

3.3 Corollary. *The canonical homomorphisms $\mathcal{A}_x \to \pi(\mathbb{A})$ are injective. We will view them as inclusions. For $e \in EA$*

$$\mathcal{A}_{o(e)} \cap e\mathcal{A}_{t(e)} e^{-1} = \alpha_e \mathcal{A}_e$$

holds in $\pi(\mathbb{A})$.

Proof. If $1 \neq g \in \mathcal{A}_x$, then (g) is a reduced π-path. By Theorem 3 $g = |(g)| \neq 1$ in $\pi(\mathbb{A})$. Thus $\mathrm{kern}(\mathcal{A}_x \to \pi(\mathbb{A})) = \{1\}$ and the homomorphism is injective.

The second assertion will be proved in two steps. One inclusion is immediate. $s \in \mathcal{A}_e \Rightarrow \alpha_e(s) \in \mathcal{A}_{\mathrm{o}(e)}$ by definition. Also $\alpha_e(s) = e\alpha_{\overline{e}}(s)e^{-1} \in e\alpha_{\overline{e}}(\mathcal{A}_e)e^{-1} \subset e\mathcal{A}_{\mathrm{t}(e)}e^{-1}$ by equation (3.1).

Conversely we have $g \in \mathcal{A}_{\mathrm{o}(e)} \cap e\mathcal{A}_{\mathrm{t}(e)}e^{-1}$, hence $|(h)| = g$ for a π-path (h) and $h \in \mathcal{A}_{\mathrm{o}(e)}$ and at the same time $|(1, e, h', \overline{e}, 1)| = g$ for some $h' \in \mathcal{A}_{\mathrm{t}(e)}$. Hence $h = |(h)| = |(1, e, h', \overline{e}, 1)| = eh'e^{-1}$ gives $eh'e^{-1}h^{-1} = 1$. By Theorem 3 $(1, e, h', \overline{e}, h^{-1})$ is not reduced, whence $h' \in \alpha_{\overline{e}}(\mathcal{A}_e)$, i.e. $h' = \alpha_{\overline{e}}(s)$ for a $s \in \mathcal{A}_e$ and $g = |(1, e, h', \overline{e}, 1)| = e\alpha_{\overline{e}}(s)e^{-1} = \alpha_e(s) \in \alpha_e(\mathcal{A}_e)$. □

3.4 Definition (S-normalized paths). For each $e \in \mathrm{E}A$, $\mathrm{o}(e) = x$, choose a set $S_e \subset \mathcal{A}_x$ of coset representatives for $\mathcal{A}_{x/e} = \mathcal{A}_x/\alpha_e\mathcal{A}_e$ so that $1 \in S_e$. Relative to that choice we call a π-path γ an *S-normalized path* if it has the form $\gamma = (s_1, e_1, \ldots, s_n, e_n, g)$, where $s_i \in S_{e_i}(1 \leq i \leq n), g \in \mathcal{A}_{\mathrm{t}(e_n)}$ and either $n = 0$ or $n > 0$ and γ is reduced, i.e. $e_{i+1} = \overline{e_i} \Rightarrow s_{i+1} \neq 1$.

3.5 Corollary (H.Bass [7]).

Two reduced π-paths γ, γ' satisfying $|\gamma| = |\gamma'|$ have the same length.

3.6 Corollary (H.Bass [7]).

For $x, y \in VA$, every element of $\pi[x, y]$ is represented by a unique S-normalized π-path from x to y.

Corollaries 3.6 and 3.5 allow us to transfer the definition of a length from π-paths to elements of the path group $\pi(\mathbb{A})$. Given two vertices $x, y \in \mathrm{V}A$ we define

$$\mathrm{len} : \pi[x, y] \to \mathbb{N}_0$$

for elements $g \in \pi[x, y]$ as equal to the length of any reduced π-path γ with $g = |\gamma|$ if $g \neq 1$ or if $g = 1$ as zero. Additionally we put

$$\pi[x, y]_n := \{g \in \pi[x, y] : \mathrm{length}(g) = n\}.$$

3.7 Lemma. *If $(x, y, n) \neq (x', y', n')$ for $(x, y, n), (x', y', n') \in VA \times VA \times \mathbb{N}_0$ then*

$$\pi[x, y]_n \cap \pi[x', y']_{n'} = \emptyset.$$

Proof. First we show, that $\pi[x, y] \cap \pi[x', y'] = \emptyset$ for $(x, y) \neq (x', y')$. By Corollary 3.6, an element of the path group $g \in \pi[x, y] \cap \pi[x', y']$ is represented by a unique S-normalized from x to y, which is also a S-normalized path from x' to y'. Hence $x = x'$ and $y = y'$.

It is sufficient now to show for every fixed pair (x, y) of vertices in A, that $\pi[x, y]_n \cap \pi[x, y]_m = \emptyset$ whenever $m \neq n$. By the same argument as above, $g \in \pi[x, y]_n \cap \pi[x, y]_m$ would be represented by an S-normalized path, which has length n and length m, hence $m = n$. □

3.8 Definition. The *fundamental group* of $\mathbb{A}$ with respect to a *base point* $x_0 \in \mathrm{V}A$ is defined as

$$\pi_1(\mathbb{A}, x_0) := \pi[x_0, x_0].$$

3.2 The universal cover

In this section the symbol G stands for the fundamental group $\pi_1(\mathbb{A}, x_0)$.

The *universal cover* of a graph of groups $\mathbb{A} = (A, \mathcal{A})$ based at $x_0 \in \mathrm{V}A$ is the graph $\mathcal{T} = \widetilde{(\mathbb{A}, x_0)}$ with vertices

$$\mathrm{V}\mathcal{T} = \bigsqcup_{x \in \mathrm{V}A} \pi[x_0, x]/\mathcal{A}_x. \tag{3.2}$$

The union is disjoint by Lemma 3.7. By Corollary 3.6 these vertices can be written uniquely in the form

$$s_1 e_1 \cdot \ldots \cdot s_n e_n \mathcal{A}_x \tag{3.3}$$

for an S-normalized π-path $(s_1, e_1, \ldots, s_n, e_n, 1)$ from x_0 to some $x \in \mathrm{V}A$ (for $n = 0$ the vertex takes the form $\mathcal{A}_{x_0}$). To simplify notation, we write $[\gamma]_x$ or $[g]_x$ for the vertex $g\mathcal{A}_x \in \pi[x_0, x]/\mathcal{A}_x$, where $g = |\gamma|$.

We make $\mathcal{T}$ a combinatorial graph (cf. Section1.4.2) by specifying edges as ordered pairs of distinct vertices. Given two vertices $[g]_x$ and $[h]_y$ we have $g \in \pi[x_0, x]$ and $h \in \pi[x_0, y]$, hence $g^{-1}h \in \pi[x, y]$ and we define

$$([g]_x, [h]_y) \in \mathrm{E}\mathcal{T} :\Longleftrightarrow g^{-1}h \in \pi[x, y]_1, \tag{3.4}$$

i.e. $g^{-1}h = set$ for some $e \in \mathrm{E}A$, $\mathrm{o}(e) = x$, $\mathrm{t}(e) = y$ and $s \in \mathcal{A}_x$, $t \in \mathcal{A}_y$, since the elements of $\pi[x, y]_1$ are represented by S-normalized paths (s, e, t) from x to y. This definition is independent of the choice of $g \in [g]_x$ respectively the choice of $h \in [h]_y$, as two different representatives differ only by an element of $\mathcal{A}_x$ respectively by an element of $\mathcal{A}_y$ multiplied from the right hand side.

The borders of an edge of $\mathcal{T}$ are really distinct. For $[g]_x = [h]_y$ we have by Lemma 3.7 $x = y$. Then $g\mathcal{A}_x = h\mathcal{A}_x$ gives $h = gs$ for $s \in \mathcal{A}_x$, thus $g^{-1}h = s \in \mathcal{A}_x \subset \pi[x, x]_0$, which has an empty intersection with $\pi[x, x]_1$ in disagreement with equation (3.4).

As usually for a combinatorial graph we define for an edge $\mathcal{E} = ([g]_x, [h]_y)$ the inverse edge $\overline{\mathcal{E}} = ([h]_y, [g]_x)$, the origin $\mathrm{o}(\mathcal{E}) = [g]_x$ and the terminus $\mathrm{t}(\mathcal{E}) = [h]_y$.

Theorem 4 (H.Bass [7]). $\mathcal{T} = \widetilde{(\mathbb{A}, x_0)}$ *is a tree.*

3.9 Remark and Definition. Bass argues in [7], Remark 1.18, that the tree $\mathcal{T} = \widetilde{(\mathbb{A}, x_0)}$ depends up to an isomorphism over A only on the graph A and the cardinality of the sets S_e chosen for an S-normalization (cf. Corollary 3.4)

$$|(S_e)| = |\mathcal{A}_{\mathrm{o}(e)/e}| = \mathrm{i}(e)$$

and does not depend on the explicit form of the groups of $\mathbb{A} = (A, \mathcal{A})$. We can thus also write $\mathcal{T} = \widetilde{(A, \mathrm{i}, x_0)}$ for *the universal cover of an edge-indexed graph.*

3.3 Group actions on the universal cover

3.3.1 Action of the fundamental group

There is a natural left action of the fundamental group $G = \pi_1(\mathbb{A}, x_0)$ on the vertices of $\mathcal{T} = \widetilde{(\mathbb{A}, x_0)}$, since for $h \in G$ and $[g]_x \in \mathrm{V}\mathcal{T}$ $h[g]_x = hg\mathcal{A}_x = [hg]_x$ holds. The quotient map is given by

$$\pi : \begin{array}{ccc} \mathrm{V}\mathcal{T} & \longrightarrow & G\backslash \mathrm{V}\mathcal{T} \\ [g]_x & \longmapsto & G[g]_x. \end{array}$$

This left action on the vertices extends for $u \in G$ and edges $\mathcal{E} = ([g]_x, [h]_y)$ to a left action on the whole graph $\mathcal{T}$ by

$$u : ([g]_x, [h]_y) \longmapsto (u[g]_x, u[h]_y).$$

To verify that this defines an automorphism, it is sufficient to show that u maps pairs of adjacent vertices to pairs of adjacent vertices. Then u defines a unique endomorphism of $\mathcal{T}$ (cf. Lemma 1.19) and the group structure of G provides an inverse function, so that G acts by automorphisms. Let us suppose, that $g^{-1}h = set$ for $s \in \mathcal{A}_{\mathrm{o}(e)}, t \in \mathcal{A}_{\mathrm{t}(e)}$. Then

$$\begin{array}{rcl} (ug)^{-1}(uh) & = & g^{-1}u^{-1}uh \\ = \quad g^{-1}h & = & set \end{array} \tag{3.5}$$

and therefore $u\mathcal{E} = (u[g]_x, u[h]_y) = ([ug]_x, [uh]_y) \in \mathrm{E}\mathcal{T}$. The orbit of an edge $\mathcal{E} \in \mathrm{E}\mathcal{T}$ is given by $G\mathcal{E}$, the quotient map π now extends to the edges by

$$\pi : \begin{array}{ccc} \mathrm{E}\mathcal{T} & \longrightarrow & G\backslash \mathrm{E}\mathcal{T} \\ \mathcal{E} & \longmapsto & G\mathcal{E} \end{array}$$

For a given edge $\mathcal{E} = ([g]_x, [h]_y) \in \mathrm{E}\mathcal{T}$ the edge $e \in \mathrm{E}A$ satisfying $g^{-1}h = set$ is unique. If there is an edge e' satisfying $g^{-1}h = set = s'e't'$, then by Theorem 3 the π-path $(s', e', t't^{-1}, \overline{e}, s^{-1})$ is not reduced, hence $\overline{e} = \overline{e'}$ implies $e' = e$. Therefore we can associate to every edge $\mathcal{E} \in \mathrm{E}\mathcal{T}$ a unique edge $e \in \mathrm{E}A$ and call this edge the *associated edge*.

It follows then, that for $u \in G$ and $\mathcal{E}$ as above, the inverse edge $\overline{\mathcal{E}}$ is associated to $\overline{e}$ while the edge $u\mathcal{E}$ is associated to the edge $e \neq \overline{e}$. Therefore the group G acts on $\mathcal{T}$ without inversions and we can form a quotient graph

$$G \backslash \mathcal{T},$$

where the graph maps are given by $\mathrm{o}\,(G\mathcal{E}) = G[g]_x$ and by $\overline{G\mathcal{E}} = G\overline{\mathcal{E}}$ and where the projection π is a graph morphism (cf. Section 1.5.1).

We can write a function $F : G\backslash\mathcal{T} \longrightarrow A$, for vertices $[g]_x$ and edges $\mathcal{E}$ by $F\,(G[g]_x) = x$ and $F\,(G\mathcal{E}) = e$, where e is the edge associated to any of the edges in $G\mathcal{E}$. By equation (3.5) these edges have all the same associated edge. We will prove that F is an isomorphism. Then we can interpreted the projection $\pi : \mathcal{T} \to G\backslash\mathcal{T}$ as a morphism $\mathcal{T} \to A$ and obtain

$$G\backslash\mathcal{T} = A. \tag{3.6}$$

Since $G = \pi[x_0, x_0] = \pi[x_0, x]|\gamma|$ for any π-path γ from x to x_0, the orbit of a vertex $[g]_x$ is give by

$$\begin{aligned} G[g]_x &= \{h[g]_x : g \in G\} = \{hg\mathcal{A}_x : h \in G\} \\ &= \{k\mathcal{A}_x : k \in \pi[x_0, x]\} = \pi[x_0, x]/\mathcal{A}_x. \end{aligned}$$

As the sets $\pi[x_0, x]$ are disjoint and by connection of A also non-empty, the function F is a bijection from the vertices of $G\backslash\mathcal{T}$ to the vertices of A (cf. equation (3.2)).

The function F is also a bijection on the edges. For surjectivity one may assume $e \in \mathrm{E}A$, $x = \mathrm{o}(e)$ and $y = \mathrm{t}(e)$. Then there is a path $a_1, \ldots, a_n$ form x_0 to x. If we choose the π-paths $\gamma_1 = (1, a_1, 1, \ldots, 1, a_n, 1)$ and $\gamma_2 = (1, a_1, 1, \ldots, 1, a_n, 1, e, 1)$, obviously e is associated to the edge $([\gamma_1]_x, [\gamma_2]_y)$. For injectivity it suffices to show, that two edges $\mathcal{E} = ([g]_x, [h]_y)$ and $\mathcal{E}' = ([g']_{x'}, [h']_{y'})$ with the same associated edge e are in the same orbit. Suppose, that $g^{-1}h = set$ while ${g'}^{-1}h' = s'et'$ for $s, s' \in \mathcal{A}_{\mathrm{o}(e)}$ and $t, t' \in \mathcal{A}_{\mathrm{t}(e)}$. Then $x = x'$ and $y = y'$ and for $u = h'(tt')^{-1}h^{-1} \in G$ we get

$$\begin{aligned} ug &= h'(tt')^{-1}t^{-1}e^{-1}s^{-1} = h'{t'}^{-1}e^{-1}s^{-1} \\ &= g's's^{-1}, \end{aligned}$$

which is an element of $g'\mathcal{A}_x$ and

$$uh = h'(tt')^{-1},$$

which is an element of $h'\mathcal{A}_y$. This shows $\mathcal{E}' = u\mathcal{E}$.

As a final step one verifies, that F is a morphism of graphs. With notation as above

$$\mathrm{o}\left(F\left(G\mathcal{E}\right)\right) = \mathrm{o}\left(e\right) = x = F\left(G[g]_x\right) = F\left(\mathrm{o}\left(G\mathcal{E}\right)\right).$$

Since $\overline{\mathcal{E}}$ is associated with the inverse edge of e we get

$$\overline{F\left(G\mathcal{E}\right)} = \overline{e} = F\left(G\overline{\mathcal{E}}\right) = F\left(\overline{G\mathcal{E}}\right).$$

3.3.2 Stars of the universal cover

In this section we want to have a look at the properties of the local maps

$$\pi_{[g]_x} : \mathrm{St}^{\mathcal{T}}([g]_x) \longrightarrow \mathrm{St}^{A}(x).$$

for vertices $[g]_x \in \mathrm{V}\mathcal{T}$. These maps are all surjective, since for every $e \in \mathrm{E}A$ with $\mathrm{o}(e) = x$ and $y = \mathrm{t}(e)$ the tuple $([g]_x, [ge]_y)$ is an edge of $\mathrm{St}^{\mathcal{T}}([g]_x)$ projecting to e under π. In order to count the edges of $\mathrm{St}^{\mathcal{T}}([g]_x)$, which project to the same edge in $\mathrm{E}A$ we are going to use some basic group theory.

The stabilizer of a vertex $[g]_x \in \mathrm{V}\mathcal{T}$ is given by

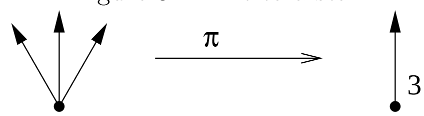
$$\begin{aligned} G_{[g]_x} &= \{h \in G : hg\mathcal{A}_x = g\mathcal{A}_x\} = \{h \in G : g^{-1}hg\mathcal{A}_x = \mathcal{A}_x\} \\ &\overset{k=g^{-1}hg}{=} g\{k \in G : k\mathcal{A}_x = \mathcal{A}_x\}g^{-1} = g\mathcal{A}_x g^{-1}. \end{aligned}$$

With notation from (3.4) for an edge $\mathcal{E} = ([g]_x, [h]_y)$ we have $[h]_y = [gset]_y = [gse]_y$. Hence the stabilizer of $\mathcal{E}$ is

$$\begin{aligned} G_{\mathcal{E}} &= G_{[g]_x} \cap G_{[h]_y} = g\mathcal{A}_x g^{-1} \cap (gse)\mathcal{A}_y (gse)^{-1} \\ &= gs(\mathcal{A}_x \cap e\mathcal{A}_y e^{-1})s^{-1}g^{-1} \overset{\text{Corollary 3.3}}{=} g(s\alpha_e\mathcal{A}_e s^{-1})g^{-1}. \end{aligned}$$

For $\mathcal{E} = ([g]_x, [h]_y) \in \mathrm{E}\mathcal{T}$, $g^{-1}h = set$, the action of $G_{[g]_x}$ on $\mathcal{T}$ restricts to an action on the edges in $\mathrm{St}^{\mathcal{T}}([g]_x)$, since $u(\mathrm{St}^{\mathcal{T}}([g]_x)) \subset \mathrm{St}^{\mathcal{T}}([g]_x)$ for all

Figure 3.2: Lift to a star



$u \in G_{[g]_x}$. The number of left cosets of the stabilizer $G_{\mathcal{E}}$ in $G_{[g]_x}$ is

$$\begin{aligned}
\left[G_{[g]_x} : G_{\mathcal{E}}\right] &= \left[g\mathcal{A}_x g^{-1} : g(s\alpha_e\mathcal{A}_e s^{-1})g^{-1}\right] \\
&\overset{\text{Lemma B.2}}{=} \left[\mathcal{A}_x : s\alpha_e\mathcal{A}_e s^{-1}\right] \\
&\overset{\text{Lemma B.1}}{=} \left[s\mathcal{A}_x s^{-1} : s\alpha_e\mathcal{A}_e s^{-1}\right] \\
&\overset{\text{Lemma B.2}}{=} \left[\mathcal{A}_x : \alpha_e\mathcal{A}_e\right] \text{ in } G \\
&\overset{\substack{\text{Corollary 3.3} \\ \mathcal{A}_x \to G \text{ is injective}}}{=} \left[\mathcal{A}_x : \alpha_e\mathcal{A}_e\right] \text{ in } \mathcal{A}_x \\
&= \mathrm{i}_A(e).
\end{aligned}$$

By Proposition I, 5.1 in [17], the size of the orbit of $\mathcal{E}$ in $\mathrm{St}^{\mathcal{T}}([g]_x)$ is $\mathrm{i}_A(e)$ and we obtain for all edges $\mathcal{E} \in \pi^{-1}(e)$

$$|\pi^{-1}_{\mathrm{o}(\mathcal{E})}(e)| = |G_{\mathrm{o}(\mathcal{E})}\mathcal{E}| = \mathrm{i}(e). \tag{3.7}$$

The first equality holds because of $G_{\mathrm{o}(\mathcal{E})}\mathcal{E} = \mathrm{St}^{\mathcal{T}}(\mathrm{o}(\mathcal{E})) \cap G\mathcal{E} = \pi^{-1}_{\mathrm{o}(\mathcal{E})}(e)$. Figure 3.2 illustrates this situation for an edge of index three. We can assemble these disjoint orbits at a star in $\mathcal{T}$ to get the result, that for every vertex $x \in \mathrm{V}\mathcal{T}$

$$|\mathrm{St}^{\mathcal{T}}(x)| = |\{\mathcal{E} \in \mathrm{E}\mathcal{T} : \mathrm{o}(\mathcal{E}) = x\}| = \sum_{\substack{e \in \mathrm{E}A \\ \mathrm{o}(e) = \pi x}} \mathrm{i}(e). \tag{3.8}$$

In particular, the tree $\mathcal{T}$ is locally finite and the vertices assume degrees bounded from above by $\mathrm{D}_{\max} := \max_{x \in \mathrm{V}A} \{ \sum_{\substack{e \in \mathrm{E}A \\ \mathrm{o}(e) = x}} \mathrm{i}(e) \}$.

3.4 The isometry group of a universal cover

We discuss the group $\mathrm{Aut}(\mathcal{T})$ of automorphisms of the universal cover to a finite connected edge indexed graph (A, i_A). Fixing a base point $x_0 \in \mathrm{V}A$, we know that the fundamental group G acts on $\mathcal{T}$ with quotient graph $A = G\backslash\mathcal{T}$. Since $\mathcal{T}$ is locally finite by equation (3.8), Corollary 1.35 gives the result $\mathrm{Is}(\mathcal{T}) = \mathrm{Aut}(\mathcal{T})$, which justifies this section's headline.

3.4.1 Subgroups of the isometry group

We shall focus on the fundamental group $G < \mathrm{Is}\,(\mathcal{T})$. More precisely, we write G for the subgroup of $\mathrm{Is}\,(\mathcal{T})$ which is the domain of the homomorphism $G \to \mathrm{Is}\,(\mathcal{T})$ of the action of the fundamental group on $\mathcal{T}$. Bass and Kulkarni argue in [11], Chapter 2, that there is always a graph of groups, called the *effective quotient*, producing an injective homomorphism from the fundamental group to the group $\mathrm{Is}\,(\mathcal{T})$. So we can think of G both as the abstract fundamental group or as a subgroup of $\mathrm{Is}\,(\mathcal{T})$.

We define the *full group* (also called the group of deck transformations)

$$G_f := \{g \in \mathrm{Is}\,(\mathcal{T}) : \pi \circ g = \pi\}.$$

G_f is indeed a group. For $g, h \in G_f$ we get $\pi \circ g = \pi \Leftrightarrow \pi = \pi \circ g^{-1}$ because the inverse exists and both g and g^{-1} are surjective maps. Also $\pi \circ (gh) = \pi \circ g \circ h = \pi \circ h = \pi$.

3.10 Lemma. *$G_F \backslash \mathcal{T} = G \backslash \mathcal{T}$, i.e. the full group has the same orbits on $\mathcal{T}$ as the fundamental group.*

Proof. For a vertex $x \in \mathrm{V}\mathcal{T}$ it is clear, that $Gx \subset G_f x$, since trivially $G \subset G_f$. Conversely, if $y \in G_f x$, then $y = gx$ for some $g \in G_f$. Since $\pi(y) = \pi \circ g(x) = \pi(x)$, there is $h \in G$ with $y = hx$. This shows $G_f x \subset Gx$. Exactly the same argumentation works for edges. $\square$

3.11 Lemma. *Suppose $e_1, e_2 \in \mathrm{E}\mathcal{T}$, $\mathrm{o}(e_1) = \mathrm{o}(e_2)$ and $\pi(e_1) = \pi(e_2)$. Then there is an isometry $h \in G_f$, such that $e_2 = he_1$ and h is the identity on the connected component of $\mathrm{o}(e_1)$ in the subgraph $\mathcal{T}'$ of $\mathcal{T}$ with edges $\mathrm{E}\mathcal{T}' = \mathrm{E}\mathcal{T} \setminus E_R$ where $E_R = \{e_1, \overline{e_1}, e_2, \overline{e_2}\}$.*

Proof. The statement is trivial for $e_1 = e_2$ choosing $h = \mathrm{Id}|_{\mathcal{T}}$. We shall suppose thus $e_1 \neq e_2$. Lemma 1.15 sums up basic arguments with connected components. We take $x = \mathrm{o}(e_1)$, $y_1 = \mathrm{t}(e_1)$ and $y_2 = \mathrm{t}(e_2)$. The graph $\mathcal{C}(y_1)$ shall be the connected component at y_1 in $\mathcal{T}'$, similarly $\mathcal{C}(y_2)$ and $\mathcal{C}(x)$ are defined. These three subgraphs cover all vertices of $\mathcal{T}$ disjointly:

The vertex x is joined to any vertex z in $\mathcal{T}$ by the reduced path $[x, z]$. If the first edge of $[x, z]$ is e_1 resp. e_2, then $[x, z] = [x, y_1][y_1, z]$ resp. $[x, z] =$

$[x, y_2][y_2, z]$, so the second path in this composition has no edge with border x by injectivity of reduced paths in trees, thus no edge of E_R. Therefore $[y_1, z]$ resp. $[y_2, z]$ is a path in $\mathcal{T}'$ proving $z \in \mathcal{C}(y_1)$ resp. $z \in \mathcal{C}(y_2)$. If the first edge of $[x, z]$ is neither e_1 nor e_2, then $[x, z]$ has no edges of E_R by injectivity of reduced paths in trees. This shows, that $[x, z]$ is a path in $\mathcal{T}'$ hence $z \in \mathcal{C}(x)$.

For disjointness note first, that $\mathcal{C}(y_1) \cap \mathcal{C}(y_2) = \emptyset$ (for $e_1 \neq e_2$). If $\mathcal{C}(y_1) = \mathcal{C}(y_2)$, there is a reduced path p from y_1 to y_2 in $\mathcal{T}'$. Since $\mathcal{T}'$ is a subgraph of $\mathcal{T}$, one has $p = [y_1, y_2] = (e_1, \overline{e_2})$, which is a contradiction to the definition of $\mathcal{T}'$ having no edges of E_R. The components $\mathcal{C}(x)$ and $\mathcal{C}(y_1)$ as well as $\mathcal{C}(x)$ and $\mathcal{C}(y_2)$ are disjoint by a similar argument about uniqueness of reduced paths.

Using this partition of $\mathcal{T}'$, we can now construct an isometry h on $\mathcal{T}$ with the required properties. By assumption there is an isometry g in the fundamental group, such that $e_2 = g(e_1)$. For any $z \in \mathcal{C}(y_1)$ we can calculate $[x, gz] = [gx, gz] = g[x, z] = g([x, y_1][y_1, z]) = [x, y_2][y_2, gz]$ which shows that $gz \in \mathcal{C}(y_2)$, that is $g(\mathcal{C}(y_1)) \subset \mathcal{C}(y_2)$. The map g as an automorphism of $\mathcal{T}$ is bijective, hence locally bijective. The graphs $\mathcal{C}(y_1)$ and $\mathcal{C}(y_2)$ are both trees as connected subgraphs of $\mathcal{T}$. Hence g defines an isomorphism $g : \mathcal{C}(y_1) \to \mathcal{C}(y_2)$ (cf. Section 1.4.3) and g^{-1} an isomorphism in the opposite direction.

Thus we define an automorphism $h|_{\mathcal{T}'}$ as follows:

$$h(z) := \begin{cases} z & \text{for all} \quad z \in \mathcal{C}(x) \\ g(z) & \text{for all} \quad z \in \mathcal{C}(y_1) \\ g^{-1}(z) & \text{for all} \quad z \in \mathcal{C}(y_2). \end{cases}$$

Obviously, $h|_{\mathcal{T}'}$ extends to an automorphism $h|_{\mathcal{T}}$ by the assignment $h(e_1) := e_2$. Since h is defined completely in terms of $\mathrm{Id}|_{\mathcal{T}}, g$ and g^{-1}, which are elements of G, one has $\pi \circ h = \pi$, thus $h \in G_f$. □

3.4.2 Topology on the isometry group

There can be defined a topology on $\mathrm{Is}\,(\mathcal{T})$ which is locally compact. The isometry group is a topological group with this topology. We follow [12] for the definition and for many arguments. All necessary terms are introduced in Appendix D. We end with statements about the subgroup G_f.

$$U_F(g) := \{h \in \mathrm{Is}\,(\mathcal{T}) : g(x) = h(x) \text{ for all } x \in F\}$$

defined for all $g \in \mathrm{Is}(\mathcal{T})$ and all finite sets $F \subset \mathrm{V}\mathcal{T}$ is a basis for a topology. We have to show that the intersection of two such sets is a neighborhood of all its elements. Suppose $h \in U_F(g) \cap U_{F'}(g')$. Then

$$\begin{aligned} U_{F\cup F'}(h) &= \{h' \in \mathrm{Is}(\mathcal{T}) : h'(x) = h(x) \text{ for all } x \in F \cup F'\} \\ &\subset \{h' \in \mathrm{Is}(\mathcal{T}) : h'(x) = h(x) \text{ for all } x \in F\} \\ &= U_F(h) = U_F(g) \end{aligned}$$

Analogously $U_{F\cup F'}(h) \subset U_{F'}(g')$, therefore $U_{F\cup F'}(h) \subset U_F(g) \cap U_{F'}(g')$. The topological space defined by this base will be written as $(\mathrm{Is}(\mathcal{T}), iso)$.

$\mathrm{Is}(\mathcal{T})$ is a topological group, with this topology, i.e. the group operations are continuous. We show continuity for the map $\beta : g \mapsto g^{-1}$ first. It is sufficient to show that $\beta^{-1}U_F(g)$ is open. Suppose $x \in F$. Then $h(x) = g(x) \Leftrightarrow h^{-1}g(x) = x$ and therefore

$$\begin{aligned} \beta^{-1}U_F(g) &= \{h^{-1} \in \mathrm{Is}(\mathcal{T}) : h(x) = g(x) \text{ for all } x \in F\} \\ &\overset{(y=gx)}{=} \{h^{-1} \in \mathrm{Is}(\mathcal{T}) : h^{-1}(y) = g^{-1}(y) \text{ for all } y \in gF\} \\ &= U_{gF}(g^{-1}). \end{aligned}$$

The map $\alpha : (g, h) \mapsto gh$ is continuous, too. Given a pair (k, l) mapping to $U_F(g)$ one has $kl(x) = g(x)$ for all $x \in F$. Consequently

$$l(x) = k^{-1}g(x)$$

for all $x \in F$ and

$$k(y) = gl^{-1}(y)$$

for all $y \in lF$. In other words $(k, l) \in U_{lF}(gl^{-1}) \times U_F(k^{-1}g)$, which is open in the product topology of $\mathrm{Is}(\mathcal{T}) \times \mathrm{Is}(\mathcal{T})$ (cf. Chapter 3 in [18]). We have to show only $U_{lF}(gl^{-1}) \times U_F(k^{-1}g) \subset \alpha^{-1}U_F(g)$. Suppose $p \in U_{lF}(gl^{-1})$ and $q \in U_F(k^{-1}g)$. Then for all $x \in F$

$$\begin{aligned} \alpha(p,q)(x) &= pq(x) \overset{x\in F}{=} pk^{-1}g(x) = pl(x) \\ &\overset{l(x)\in lF}{=} gl^{-1}l(x) = g(x). \end{aligned}$$

3.12 Proposition. $(\mathrm{Is}(\mathcal{T}), iso)$ *is a topological group with the above defined base.*

We shall now prove some properties of this topological group. The vertex stabilizers $K_x = \{g \in \mathrm{Is}(\mathcal{T}) : g(x) = x\}$ are *iso*-open, since $K_x = U_{\{x\}}(\mathrm{Id})$.

We show now that they are also compact. An element $g \in K_x$ acts on each set $W_n = \{y \in \mathrm{V}\mathcal{T} : \mathrm{d}(x,y) = n\}$ as a permutation. Since $\mathcal{T}$ is locally finite, $|W_n|$ is finite for all $n \in \mathbb{N}_0$. Say $|W_n| = r_n$. We write S_k for the symmetric group of the numbers $\{1,\ldots,k\}$ endowed with the discrete topology (all subsets are open). Then the topological groups S_n are all compact. The group

$$L := S_{r_0} \times S_{r_1} \times S_{r_2} \times S_{r_3} \times \ldots$$

acts on $W_0 \cup W_1 \cup W_2 \cup \ldots$ in the obvious way and $K_x < L$. L with the product topology of the permutation groups is then by Theorem 2 in [19] (Tychonoff's Theorem) compact. We write (L, perm) for this topological space. A basis for (L, perm) can be defined by the family of sets

$$\big\{l \in L : l_n \in U_n \text{ for all } n \in E\big\},$$

(cf [18]), where l_n is the projection of l to the n^{th} coordinate. For example $(\omega_0, \omega_1, \omega_2, \omega_3, \ldots)_2 = \omega_2$. U_n is an open set, hence any subset of S_{r_n} and $E \subset \mathbb{N}_0$ is finite.

Each group element $g \in L \setminus K_x$ is not an isometry. Hence there are vertices y, z such that $\mathrm{d}(gy, gz) \neq \mathrm{d}(y,z)$. We assume that $\mathrm{d}(x,y) = n_1$ and $\mathrm{d}(x,z) = n_2$ and set

$$L' := \big\{h \in L : h_{n_1} \in \{g_{n_1}\}, h_{n_2} \in \{g_{n_2}\}\big\}.$$

L' is a base member of (L, perm), hence perm-open. Also $L' \subset L \setminus K_x$ because for $h \in L'$ we get $\mathrm{d}(hy, hz) = \mathrm{d}(h_{n_1}(y), h_{n_1}(z)) = \mathrm{d}(g_{n_1}(y), g_{n_1}(z)) = \mathrm{d}(gy, gz) \neq \mathrm{d}(y,z)$. Therefore $L \setminus K_x$ is perm-open and K_x is perm-closed. A closed subset of a compact set is compact, hence K_x is perm-compact.

We proceed by showing that the *iso*-topology of $\mathrm{Is}\,(\mathcal{T})$ is included in the relative topology inherited from (L, perm). It is sufficient to show that for each subset $C \subset \mathrm{Is}\,(\mathcal{T})$ of the form $C = U_F(g)$ with $g \in \mathrm{Is}\,(\mathcal{T})$ and $F \subset \mathrm{V}\mathcal{T}$ finite, there exists an perm-open set D satisfying $C = \mathrm{Is}\,(\mathcal{T}) \cap D$.

Given $C = U_F(g)$ there is a finite decomposition $F = V_{n_1} \cup \ldots V_{n_k}$ with $V_{n_i} \subset W_{n_i}$ for all $1 \leq i \leq k$. For each $n \in E := \{n_1, \ldots, n_k\}$ we define

$$U_n := \{h_n \in S_{r_n} : h_n(y) = g_n(y) \text{ for all } y \in V_n\}$$

and set

$$D := \{k \in L : k_n \in U_n \text{ for all } n \in E\}.$$

For $k \in C$ and any $n \in E$ one has $k_n(y) = g_n(y)$ for all $y \in V_n \subset F$, hence $k_n \in U_n$, which implies then $k \in D$; in other works $C \subset D$. It remains to show $(D \setminus C) \cap \mathrm{Is}(\mathcal{T}) = \emptyset$. For any $y \in F$ there is $n \in E$ such that $y \in V_n$. Given $k \in D$ we get $k(y) = k_n(y) = l_n(y)$ for some $l_n \in U_n$. Hence $l_n(y) = g_n(y)$ gives $k(y) = g(y)$. Under the assumption $k \in \mathrm{Is}(\mathcal{T})$ this amounts to $k \in C$.

By Lemma D.3, K_x is compact in the topology inherited from *perm*. As shown in the above paragraph, any open cover of K_x by members of the *iso*-topology is an open cover of K_x by members of the topology inherited from *perm*, so K_x is *iso*-compact.

3.13 Proposition. *The topological group* $(\mathrm{Is}(\mathcal{T}), iso)$ *is locally compact. All vertex stabilizers are open and compact.*

Proof. By Proposition 3.12, $(\mathrm{Is}(\mathcal{T}), iso)$ is a topological group. Now each vertex stabilizer is open and compact, hence a neighborhood of the identity. Proposition 16 in [19] shows then local compactness. □

3.14 Proposition. *The topological space* $(\mathrm{Is}(\mathcal{T}), iso)$ *is a Hausdorff space.*

Proof. For $g, h \in \mathrm{Is}(\mathcal{T})$ and $g \neq h$ there is a vertex $x \in \mathrm{V}\mathcal{T}$ with $g(x) \neq h(x)$ therefore $U_{\{x\}}(g)$ and $U_{\{x\}}(h)$ are disjoint *iso*-neighborhoods of g and h respectively. □

3.15 Proposition. *The group* G_f *is an iso-closed subgroup of* $\mathrm{Is}(\mathcal{T})$.

Proof. We can construct a neighborhood around each element $h \in \mathrm{Is}(\mathcal{T}) \setminus G_f$ which has empty intersection with G_f. There is essentially one[1] possible case for $\pi \circ h \neq \pi$, the existence of an edge $e \in \mathrm{E}\mathcal{T}$ with $\pi(e) \neq \pi \circ h(e)$. This is clear, because if $\pi(e) = \pi \circ h(e)$ holds for all edges, then for any vertex x there is an edge e with $\mathrm{o}(e) = x$ by connection of $\mathcal{T}$ and we find $\pi(x) = \pi(\mathrm{o}(e)) = \mathrm{o}(\pi(e)) = \mathrm{o}(\pi \circ h(e)) = \pi \circ h(\mathrm{o}(e)) = \pi \circ h(x)$.

Given an edge $e \in \mathrm{V}\mathcal{T}$ such that $\pi(he) \neq \pi(e)$ one has $h(e) \notin G_f e$. Since $\mathcal{T}$ is combinatorial, we calculate $k(e) = h(e) \neq g(e)$ for each $k \in U_{\{\mathrm{o}(e),\mathrm{t}(e)\}}(h)$

[1] The case that a subgroup of $\mathrm{Is}(\mathcal{T})$ containing G_f has the same vertex orbits on $\mathcal{T}$ as G_f but may identify more edges can happen if the quotient graph has multiple edges, i.e. more than one geometric edges share the same borders. Consider for example the regular tree of degree four, $\mathcal{T}_4$, as a cover of circ_2 with an indexing constant to two.

and for all $g \in G_f$, hence $U_{\{\mathrm{o}(e),\mathrm{t}(e)\}}(h) \cap G_f = \emptyset$. This shows, that $\mathrm{Is}\,(\mathcal{T}) \setminus G_f$ is open thus G_f is closed. □

3.16 Corollary. *The full group G_f is a locally compact topological group, which is a Hausdorff space. All vertex stabilizers are open and compact.*

Proof. We use for G_f the relative topology *full* inherited from *iso*. By Proposition 3.15 G_f is an *iso*-closed subgroup of $\mathrm{Is}\,(\mathcal{T})$, hence G_f is a locally compact topological group after Proposition 16 in [19]. A subspace of a Hausdorff space is Hausdorff in the relative topology. Proposition 3.14 is sufficient then.

For each vertex $x \in \mathrm{V}\mathcal{T}$, the stabilizer $(G_f)_x$ can be written as $G_f \cap \mathrm{Is}\,(\mathcal{T})_x$, hence is $full$-open because $\mathrm{Is}\,(\mathcal{T})_x$ is *iso*-open by Proposition 3.13. $(G_f)_x$ is an *iso*-closed subset of the *iso*-compact stabilizer $\mathrm{Is}\,(\mathcal{T})_x$, hence is *iso*-compact. By Lemma D.3 $(G_f)_x$ it is *full*-compact. □

3.5 Examples

Some explicit examples of covering trees $(\widetilde{A, \mathrm{i}_A, x_0})$ to finite connected edge-indexed graphs (A, i_A) will be given here. As introduced earlier, we write vertices in a unique way in the form $[|\gamma|]_x$ for S-normalized π-paths γ.

We may start drawing with the vertex $[1]_{x_0}$. The vertices of the form $\{[ge]_{\mathrm{t}(e)} : \mathrm{o}(e) = x_0, g \in S_e\}$ are adjacent to $[1]_{x_0}$. By equation (3.8) there are no more adjacent vertices ($\mathrm{i}(e) = |S_e|$ for edges $e \in \mathrm{E}A$).

Given a vertex $[ge]_{\mathrm{t}(e)}$ as above ($\mathrm{o}(e) = x_0$), the vertex $[1]_{x_0}$ is adjacent and the edge $([ge]_{\mathrm{t}(e)}, [1]_{x_0})$ projects to $\overline{e}$ under the quotient map π. Other adjacent vertices of $[ge]_{\mathrm{t}(e)}$ are $\{[gehe']_{\mathrm{t}(e')} : \mathrm{o}(e') = \mathrm{t}(e), h \in S_{e'}\}$, where the vertex $[ge1\overline{e}]_{x_0}$ is equal to the vertex $[1]_{x_0}$. By equation (3.8) these are all adjacent vertices.

More generally for $\mathrm{o}(e_1) = x_0$ and $n \geq 2$, the vertex corresponding to an S-normalized path $\gamma = (g_1, e_1, \ldots, g_n, e_n, 1)$ is written $[g_1e_1 \ldots g_ne_n]_{\mathrm{t}(e_n)}$. It is adjacent to the vertex $[g_1e_1 \ldots g_{n-1}e_{n-1}]_{\mathrm{t}(e_{n-1})}$. The edge $([g_1e_1 \ldots g_ne_n]_{\mathrm{t}(e_n)}, [g_1e_1 \ldots g_{n-1}e_{n-1}]_{\mathrm{t}(e_{n-1})})$ projects to $\overline{e_n}$ under π. Other adjacent vertices are $\{[g_1e_1 \ldots g_ne_nge']_{\mathrm{t}(e')} : \mathrm{o}(e') = \mathrm{t}(e_n), h \in S_{e'}\}$. $[g_1e_1 \ldots g_ne_n1\overline{e_n}]_{\mathrm{t}(\overline{e_n})}$ can be written in S-normalized form as $[g_1e_1 \ldots g_{n-1}e_{n-1}]_{\mathrm{t}(e_{n-1})}$. By equation (3.8) these are all adjacent vertices.

Since the representatives chosen for an S-normalization always precede the corresponding edge in such sequences, we write simply $\{1, \ldots, \mathrm{i}(e)\}$ instead of S_e for all edges $e \in \mathrm{E}A$. We use the convention to write the inverse of an edge $(a, b, c, \ldots)$ as the same letter in upper case $(A, B, C, \ldots)$ and conversely to write the inverse of an edge $(A, B, C, \ldots)$ by its corresponding lower case letter $(a, b, c, \ldots)$.

- For path_1 with indexing constant to one (Figure 3.3) we form two covering trees, depending on the base point $x_0 = x$ or $x_0 = y$ (Figure 3.4).

Figure 3.3: $(\mathrm{path}_1, 1)$

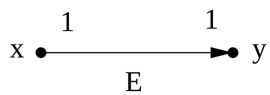


Figure 3.4: Covers of $(\mathrm{path}_1, 1)$

- We change the first example by $\mathrm{i}(e) = 2$ (Figure 3.5), and obtain two covering trees with base points x or y as shown in Figure 3.6.

Figure 3.5: path_1 indexed by 1 and 2

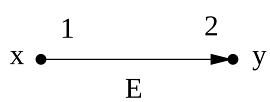
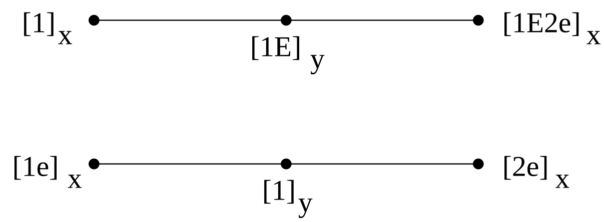


Figure 3.6: Covers to path_1 indexed by 1 and 2



- For path_1 with indices $\mathrm{i}(E) = \mathrm{i}(e) = 2$ (Figure 3.7), the universal cover $\mathcal{T}_2$ relative to the base point $x_0 = x$ can be seen in Figure 3.8.

Figure 3.7: path_1 indexed by 2

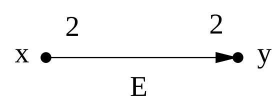
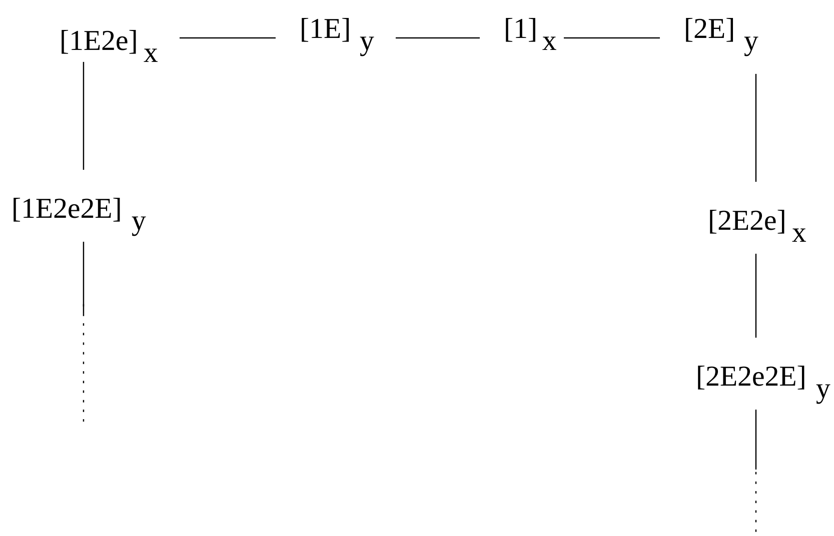


Figure 3.8: $\mathcal{T}_2$ as cover of path_1 indexed by 2



- In Figure 3.9 the universal cover to the edge-indexed graph of Figure 3.10 is drawn relatively to the base point $x_0 = w$.

Figure 3.9: A more complicated cover

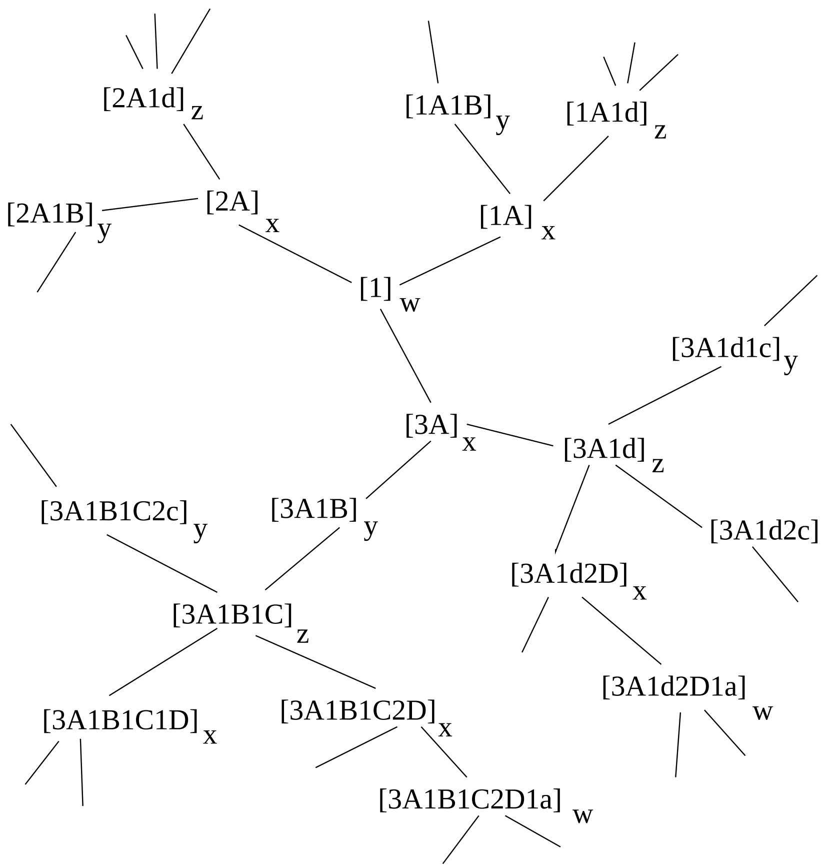
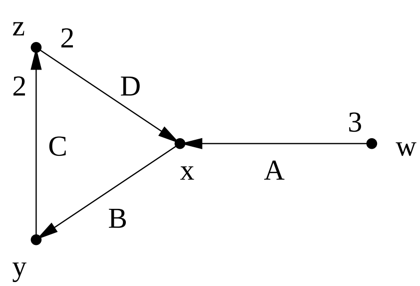


Figure 3.10: A more complicated edge-indexed graph



Chapter 4

The border of a tree

The symbol $\mathcal{T}$ is supposed to be a locally finite tree inside this chapter. The border $\mathcal{T}(\infty)$ of $\mathcal{T}$ is going to be constructed and will help us to write bi-infinite reduced rays in $\mathcal{T}$ in a very convenient way. In Part III we are going to identify bi-infinite reduced paths with two border points and an integer number. The horocycle distance from Section 4.3 will prove to be useful thereby.

4.1 Ray spaces and the abstract border

A tree is a combinatorial graph, so we are allowed to write paths by their vertex sequence. After Proposition 1.22 we can denote the unique reduced path joining x with y by $[x, y]$. A ray can be written as

$$x_0, x_1, x_2, \ldots$$

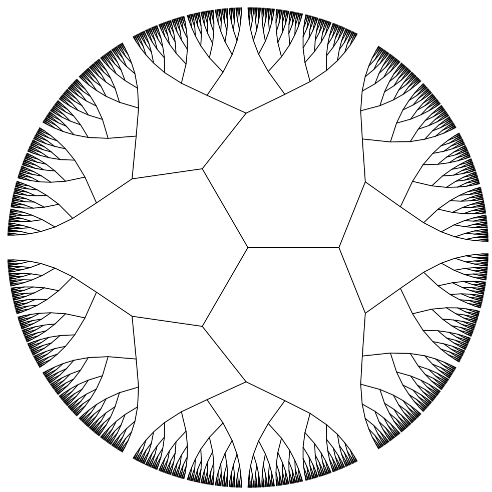
with pairwise adjacent vertices $x_i \in \mathrm{V}\mathcal{T}$. Note also, that a reversal $e, \overline{e}$ in terms of the vertex sequence is a path (x, y, z) with $x = z$. There is another consequence of uniqueness of reduced paths in trees. If the vertex sequences (x_i) and (y_j) of two reduced paths obey $x_k = y_l$ and $x_{k+1} \neq y_{l+1}$ then $\{x_i : i > k\} \cap \{y_j : j > l\} = \emptyset$. This is also true for infinite reduced rays because it holds for all their finite segments.

The *relative border* of a tree $\mathcal{T}$ with respect to a given vertex $x \in \mathrm{V}\mathcal{T}$ is the set

$$\mathcal{T}_x(\infty) = \{(x_0, x_1, x_2, \ldots) \in \mathcal{R}_\infty(\mathcal{T}) : x_0 = x\}$$

consisting of all reduced rays $(x_0, x_1, x_2, \ldots)$ with origin $x_0 = x$. Figure 4.1 gives a suggestive impression of this construction in the case of a three-regular tree.

Figure 4.1: Visualization of the border of $\mathcal{T}_3$



4.1 Definition (Border of a tree). The *border of a tree* is defined as the classes of an equivalence relation.

$$\mathcal{T}(\infty) := \mathcal{R}_\infty(\mathcal{T}) / \sim$$

Two reduced rays $(x_0, x_1, \ldots)$ and $(y_0, y_1, \ldots)$ are equivalent, if and only if they have the *infinite intersection property*, i.e. there are $k_1, k_2 \in \mathbb{N}_0$ such that $x_{k_1+l} = y_{k_2+l}$ holds for all $l \in \mathbb{N}_0$.

Reflexivity is clear by the choice $k_1 = k_2 = 0$. Symmetry is a heritage of symmetry in the definition. If $(x_0, x_1, \ldots) \sim (y_0, y_1, \ldots)$ and $(y_0, y_1, \ldots) \sim (z_0, z_1, \ldots)$, there are natural numbers k_1, k_2, k_3, k_4, such that $x_{k_1+l_1} = y_{k_2+l_1}$ and $y_{k_3+l_2} = z_{k_4+l_2}$ for all $l_1, l_2 \in \mathbb{N}_0$. Then $x_{(k_1+k_3-k_2)+l} = z_{k_4+l}$ for all $l \in \mathbb{N}_0$ if $k_2 \leq k_3$ or $x_{k_1+l} = z_{(k_4+k_2-k_3)+l}$ for all $l \in \mathbb{N}_0$ if $k_3 \leq k_2$ show transitivity.

4.2 Lemma. *For every vertex $x \in \mathrm{V}\mathcal{T}$ there is a one-to-one correspondence*

$$\begin{array}{rcl} \mathcal{T}_x(\infty) & \longrightarrow & \mathcal{T}(\infty) \\ (x_0, x_1, \ldots) & \longmapsto & [(x_0, x_1, \ldots)] \end{array}$$

sending a reduced ray from $\mathcal{T}_x(\infty) \subset \mathcal{R}_\infty(\mathcal{T})$ to its equivalence class.

Proof. To show surjectivity, suppose $\omega \in \mathcal{T}(\infty)$ and let $(y_0, y_1, \ldots) \in \omega$ be a fixed representative. The relation $\mathrm{d}(x, y_{k+1}) < \mathrm{d}(x, y_k)$ is not true for all $k \in \mathbb{N}_0$ because d as a metric assumes only positive values. The relation $\mathrm{d}(x, y_{k+1}) = \mathrm{d}(x, y_k)$ if false for all $k \in \mathbb{N}_0$ by Lemma 1.26.

As a consequence there is a pair of vertices y_k, y_{k+1} such that $\mathrm{d}(x, y_k) < \mathrm{d}(x, y_{k+1})$, which results in the contradiction $\mathrm{d}(y_k, y_{k+1}) > 1$, if the strict metric inequality $\mathrm{d}(x, y_{k+1}) < \mathrm{d}(x, y_k) + \mathrm{d}(y_k, y_{k+1})$ holds. Therefore $\mathrm{d}(x, y_{k+1}) = \mathrm{d}(x, y_k) + \mathrm{d}(y_k, y_{k+1})$. Because of this the composition $[x, x_k][x_k, x_{k+1}]$ is reduced according to Lemma 1.25 whence $[x, y_k](y_k, y_{k+1}, y_{k+2}, \ldots)$ is a reduced ray in ω starting at x.

To verify injectivity, suppose $(x_0, x_1, \ldots) \neq (y_0, y_1, \ldots)$ for two reduced rays in $\mathcal{T}_x(\infty)$. We find a smallest number $k \in \mathbb{N}_0$ with $x_k = y_k$ and $x_{k+1} \neq y_{k+1}$ because $x_0 = y_0 = x$. But then infinite intersection is impossible because $\{x_i : i > k\} \cap \{y_j : j > k\} = \emptyset$. □

We use the above correspondence to simplify notation. $[x, \omega)$ stands for the unique reduced ray with origin x in the class ω and we say $[x, \omega)$ is the reduced ray *from* the vertex x *to* the border point ω. We abbreviate also $x_\omega := [x, \omega)$, which allows us to write $(x_\omega(0), x_\omega(1), x_\omega(2) \ldots)$ for $[x, \omega)$ in the interpretation of x_ω as a morphism from path_∞ to $\mathcal{T}$.

4.3 Definition (Isometric group action on the border). If a tree $\mathcal{T}$ is the universal cover of a finite connected edge indexed graph, then the group $\mathrm{Is}(\mathcal{T})$ acts on $\mathcal{T}$ (cf. Section 3.4.1). This action extends to the set of rays in $\mathcal{T}$, since given a ray $r : \mathrm{path}_\infty \to \mathcal{T}$, then for all $g, h \in \mathrm{Is}(\mathcal{T})$ we have $(gh)r = g(hr)$ by the associativity of map composition. Also $\mathrm{Id}_{\mathcal{T}} \circ r = r$. From the description of the action by map composition we obtain particularly for reduced rays $x_\omega \in \mathcal{R}_\infty(\mathcal{T})$

$$(hx_\omega)(i) = h(x_\omega(i)) \tag{4.1}$$

for all $i \in \mathbb{N}_0$. As each isometry is injective, hx_ω is also reduced, i.e. $\mathrm{Is}(\mathcal{T})$ acts on $\mathcal{R}_\infty(\mathcal{T})$. Given two reduced rays p, q with infinite intersection, the images hp, hq clearly have infinite intersection, hence $\mathrm{Is}(\mathcal{T})$ acts on $\mathcal{T}(\infty)$. Since $hx_\omega \in h\omega$ and since $(hx_\omega)(0) = hx$ by equation (4.1) we can write

$$h(x_\omega) = (hx)_{(h\omega)}. \tag{4.2}$$

4.2 Metrics on the border of a tree

Each space of reduced rays $\mathcal{T}_x(\infty)$ carries a natural metric. Under this metric two border points $[x, \eta)$ and $[x, \xi)$ are near by, if they share a large common segment.

4.4 Definition (Intersection of rays with coinciding origin). For every vertex $x \in \mathrm{V}\mathcal{T}$ and two border points η, ξ we define

$$L^x_{\eta\xi} := \sup_{j \in \mathbb{N}_0} \{x_\eta(j) = x_\xi(j)\}.$$

L^x takes values in $\mathbb{N}_0 \cup \{\infty\}$. The intersection $[x, \eta) \cap [x, \xi)$ is defined as the segment $[x, t(x)]$ with

$$t(x) := x_\eta(L^x_{\eta\xi}) = x_\xi(L^x_{\eta\xi})$$

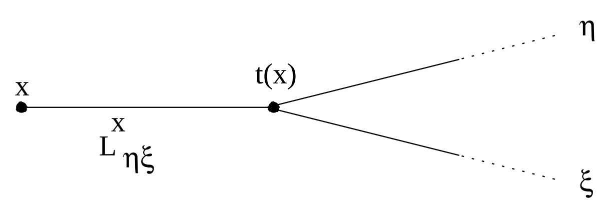
for $L^x_{\eta\xi} < \infty$. (Confer Figure 4.2.)

$$[x, \eta) \cap [x, \xi) = [x, \eta) = [x, \xi)$$

for $L^x_{\eta\xi} = \infty$.

The intersection is commutative by construction, since $L_{\eta\xi} = L_{\xi\eta}$. If there is no danger of confusion, we drop some of the indices of $L^x_{\eta\xi}$.

Figure 4.2: Intersection of rays with a common origin



4.5 Definition (Visual metrics on the border). Thus prepared we define a distance on $\mathcal{T}(\infty)$ by $\mathrm{d}_x(\eta, \xi) := e^{-L^x_{\eta\xi}}$. d_x is called the *visual metric* on $\mathcal{T}(\infty)$ *with respect to* x.

By equations (4.2) and (4.1) for all $\eta, \xi \in \mathcal{T}(\infty)$, $x \in \mathrm{V}\mathcal{T}$ and $h \in \mathrm{Is}(\mathcal{T})$ holds $(hx)_{(h\omega)}(i) = h(x_\omega)(i) = h(x_\omega(i))$ for all $i \in \mathbb{N}_0$. Hence by bijectivity of h one has $x_\eta(i) = x_\xi(i) \Leftrightarrow (hx)_{(h\eta)}(i) = (hx)_{(h\xi)}(i)$, which implies

$$\mathrm{d}_x(\eta, \xi) = \mathrm{d}_{hx}(h\eta, h\xi). \tag{4.3}$$

4.6 Lemma. *For every vertex $x \in \mathrm{V}\mathcal{T}$ the distance $\mathrm{d}_x(\eta,\xi) = e^{-L_{\eta\xi}}$ is a metric on $\mathcal{T}(\infty)$.*

Proof. We choose three border points α, β, γ and verify $\mathrm{d}_x(\alpha,\alpha) = e^{-L_{\alpha\alpha}} = e^{-\infty} = 0$, $\mathrm{d}_x(\alpha,\beta) = 0 \Rightarrow L_{\alpha\beta} = \infty \Rightarrow [x,\alpha) = [x,\beta) \Rightarrow \alpha = \beta$ and $\mathrm{d}_x(\alpha,\beta) = e^{-L_{\alpha\beta}} = e^{-L_{\beta\alpha}} = \mathrm{d}_x(\beta,\alpha)$. For transitivity we consider several cases. If $L_{\alpha\beta} \leq L_{\alpha\gamma}$ then $\mathrm{d}_x(\alpha,\gamma) = e^{-L_{\alpha\gamma}} \leq e^{-L_{\beta\gamma}} \leq e^{-L_{\alpha\beta}} + e^{-L_{\beta\gamma}} = \mathrm{d}_x(\alpha,\beta) + \mathrm{d}_x(\beta,\gamma)$. Similarly for $L_{\beta\gamma} \leq L_{\alpha\gamma}$ we find $\mathrm{d}_x(\alpha,\gamma) \leq \mathrm{d}_x(\alpha,\beta) + \mathrm{d}_x(\beta,\gamma)$. In case that $L_{\alpha\beta} > L_{\alpha\gamma}$ and $L_{\beta\gamma} > L_{\alpha\gamma}$, the ray $[x,\beta)$ shares with both $[x,\alpha)$ and $[x,\gamma)$ a segment of length $L_{\alpha\gamma}+1$. Hence $[x_\alpha(0), x_\alpha(L_{\alpha\gamma}+1)] = [x_\beta(0), x_\beta(L_{\alpha\gamma}+1)] = [x_\gamma(0), x_\gamma(L_{\alpha\gamma}+1)]$ implies $L_{\alpha\gamma} \geq L_{\alpha\gamma}+1$. □

4.7 Proposition. *The metrics d_x and d_y are equivalent for all vertices $x, y \in \mathrm{V}\mathcal{T}$. More specifically one has the bounds*

$$e^{-d(x,y)} d_x(\eta,\xi) \leq d_y(\eta,\xi) \leq e^{d(x,y)} d_x(\eta,\xi)$$

for all border points η, ξ.

Proof. We are going to show the inequality (with notation as in Definition 4.4)

$$\mathrm{d}(y, t(y)) \geq -\mathrm{d}(x,y) + \mathrm{d}(x, t(x))$$

for all vertices x, y, which gives the right inequality of the statement. The left one comes from interchanging x and y.

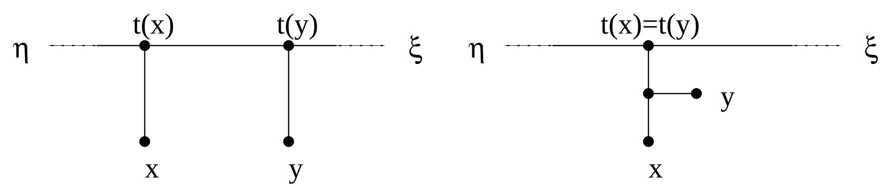
As a main step we show that the path $[x, t(x)][t(x), t(y)]$ is reduced for all vertices $x, y \in \mathrm{V}\mathcal{T}$ (similarly $[y, t(y)][t(y), t(x)]$ is reduced). If this was not the case, then $[t(y), t(x)][t(x), \eta) = [t(y), \eta)$ as well as $[t(y), t(x)][t(x), \xi) = [t(y), \xi)$. Since both $[y, t(y)][t(y), \eta) = [y, \eta)$ and $[y, t(y)][t(y), \xi) = [y, \xi)$ hold, one obtains

$$\begin{aligned} & [y, t(y)][t(y), t(x)][t(x), \eta) = [y, \eta) \\ \text{and} \quad & [y, t(y)][t(y), t(x)][t(x), \xi) = [y, \xi). \end{aligned}$$

Additionally, since $[x, t(x)][t(x), t(y)]$ was supposed not to be reduced, one can assume the length of $[t(x), t(y)]$ to be positive. This is a contradiction to the definition of $t(y)$ as the vertex of $[y, \eta)$ and $[y, \xi)$ most far away from y.

The following two cases basically correspond to the two situations indicated in Figure 4.3. If $[t(x), t(y)]$ has positive length, then $[x, t(x)][t(x), t(y)][t(y), y]$

Figure 4.3: Different ways of intersections (up to symmetry)



is reduced, since its partial compositions are. Then by Lemma 1.25 $\mathrm{d}(x,y) = \mathrm{d}(x,t(x)) + \mathrm{d}(t(x),t(y)) + \mathrm{d}(t(y),y) \geq \mathrm{d}(x,t(x)) - \mathrm{d}(y,t(y))$. Otherwise $t(x) = t(y)$ and $\mathrm{d}(x,t(x)) \leq \mathrm{d}(x,y) + \mathrm{d}(y,t(x)) = \mathrm{d}(x,y) + \mathrm{d}(y,t(y))$. □

This property is a very beneficial one, since it implies, that the induced metric topologies on the border will all be the same. We will come back to topology in Section 4.4.

4.8 Lemma. *The metric space $(\mathcal{T}(\infty), \mathrm{d}_x)$ is a complete metric space for all $x \in \mathrm{V}\mathcal{T}$.*

Proof. Assume $(\xi_j)_{j\in\mathbb{N}}$ is a Cauchy-sequence with respect to the metric d_x. Then by definition for all $\epsilon > 0$ there is a $s(\epsilon) \in \mathbb{N}$, such that for all natural numbers $m, n \geq s(\epsilon)$ $\mathrm{d}_x(\xi_m, \xi_n) < \epsilon$. Hence for all $l \in \mathbb{N}_0$ there is $t(l) \in \mathbb{N}$, such that for all $m, n \geq t(l)$ the length of the ray $[x, \xi_m) \cap [x, \xi_n)$ is greater equal that l. We write X_j for $[x, \xi_j)$ and can restate above property as: For all $l \in \mathbb{N}_0$ there is $t(l) \in \mathbb{N}_0$ such that for all $m, n \geq t(l)$

$$[X_n(0), X_n(l)] = [X_m(0), X_m(l)]$$

We define a vertex sequence by

$$Y(k) = X_{t(k)}(k) \quad \text{for } k \in \mathbb{N}_0.$$

For each finite vertex sequence $X_{t(0)}(0), \ldots, X_{t(l)}(l)$ we choose $T(l) := \max\limits_{0\leq i\leq l} t(l)$. Then by assumption $Y(i) = X_{t(i)}(i) = X_{T(i)}(i)$ for all $0 \leq i \leq l$, hence

$$[Y(0), Y(l)] = [X_{T(l)}(0), X_{T(l)}(l)],$$

thus Y is reduced. Now for the border point $\omega \in \mathcal{T}(\infty)$ with $Y \in \omega$ we get $L_{\omega,\xi_n} = \mathrm{len}(Y \cap X_n) \geq l$ for all $n \geq T(l)$, i.e. $\mathrm{d}_x(\omega, \xi_n) \leq e^{-l}$ for all $n \geq T(l)$ or equivalently $\lim\limits_{j\to\infty} \xi_j = \omega$. □

4.9 Proposition. *Each sequence $(\xi_n)_{n\in\mathbb{N}}$ in the metric space $(\mathcal{T}(\infty), \mathrm{d}_x)$ has a subsequence converging to an element of $\mathcal{T}(\infty)$.*

Proof. We use a method called *diagonal process*. We introduce $W_n := \{y \in \mathrm{V}\mathcal{T} : \mathrm{d}(x,y) = n\}$ for $n \in \mathbb{N}_0$. These are all finite sets because $\mathcal{T}$ is locally finite. Suppose $X_{0,0}, X_{0,1}, X_{0,2}, \ldots$ is the sequence of rays in $\mathcal{T}_x(\infty)$ corresponding to the border points $\xi_0, \xi_1, \xi_2, \ldots$, all of which have the common origin x.

We can choose a subsequence $(X_{1,i})_{i\in\mathbb{N}_0} \subset (X_{0,i})_{i\in\mathbb{N}_0}$, which contains only rays that coincide in their second vertex $X_{1,i}(1)$. The choice of such a subsequence is possible because by finiteness of $W_1 = \{y \in \mathrm{V}\mathcal{T} : \mathrm{d}(x,y) = 1\}$ at least one vertex $y \in W_1$ pertains to infinitely many of the rays $(X_{0,i})_{i\in\mathbb{N}_0}$. By finiteness of W_2 we can choose a subsequence of rays $(X_{2,i})_{i\in\mathbb{N}_0} \subset (X_{1,i})_{i\in\mathbb{N}_0}$ sharing the third vertex $X_{2,i}(2)$. This selection of subsequences can be continued inductively by choosing $(X_{n,i})_{i\in\mathbb{N}_0} \subset (X_{n-1,i})_{i\in\mathbb{N}_0}$ with equal vertex $X_{n,i}(n) \in W_n$ and produce

$$\begin{array}{ccccc} X_{0,0}, & X_{0,1}, & X_{0,2}, & X_{0,3}, & \ldots \\ X_{1,0}, & X_{1,1}, & X_{1,2}, & X_{1,3}, & \ldots \\ X_{2,0}, & X_{2,1}, & X_{2,2}, & X_{2,3}, & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{array}.$$

The sequence $(\eta_j)_{j\in\mathbb{N}_0}$ defined by $X_{j,j} = [x, \eta_j)$ for $j \in \mathbb{N}_0$ is a subsequence of $(\xi_j)_{j\in\mathbb{N}_0}$. Choosing $k \in \mathbb{N}_0$, the inclusions

$$(X_{0,j})_{j\in\mathbb{N}_0} \supset (X_{1,j})_{j\in\mathbb{N}_0} \supset (X_{2,j})_{j\in\mathbb{N}_0} \supset \cdots$$

imply that $[X_{n,i}(0), X_{n,i}(k)] = [X_{k,j}(0), X_{k,j}(k)]$ for all $i, j \in \mathbb{N}_0$ and all $n \geq k$. Therefore, in particular $L_{\eta_m\eta_n} \geq k$ for $m, n \geq k$. This shows $\mathrm{d}_x(\eta_m, \eta_n) \leq e^{-k}$ for all $m, n \geq k$ and proves, that $(\eta_j)_{j\in\mathbb{N}_0}$ is a Cauchy-sequence. Since the metric space $(\mathcal{T}(\infty), \mathrm{d}_x)$ is complete (cf. Lemma 4.8), this sequence converges. □

4.3 Horocycles of a tree

4.10 Definition (Projection to a ray). Given a border point $\omega \in \mathcal{T}(\infty)$ and two vertices $x, z \in \mathrm{V}\mathcal{T}$, we define the *projection* $\mathbf{p}(x, z, \omega)$ of x to $[z, \omega)$ as the unique vertex

$$\mathbf{p}(x, z, \omega) \in [z, \omega) \quad \text{such that} \quad \mathrm{d}(x, \mathbf{p}(x, z, \omega)) = \min_{z' \in [z,\omega)} \mathrm{d}(x, z'). \tag{4.4}$$

We have to verify that this definition makes sense. The existence of the projection is clear by well-ordering of $\mathbb{N}$. If p, q are two candidate for $\mathbf{p}(x, z, \omega)$, then both $[x, p][p, q]$ and $[x, q][q, p]$ are reduced, since $[p, q] \subset [z, \omega)$ and a projection is a vertex of $[z, \omega)$ closest to x. Thus by Lemma 1.25 one has

$$\begin{aligned} \mathrm{d}(x,p) + \mathrm{d}(p,q) &= \mathrm{d}(x,q) \quad \text{and} \\ \mathrm{d}(x,q) + \mathrm{d}(q,p) &= \mathrm{d}(x,p). \end{aligned}$$

The sum of these two equations gives $\mathrm{d}(p, q) = 0$ and shows $p = q$.

Observe that the ray $[x, \mathbf{p}(x, z, \omega)][\mathbf{p}(x, z, \omega), \omega)$ is reduced. We may put $[x, \mathbf{p}(x, z, \omega)] := (x_0, \ldots, x_k)$ as vertex sequence. Remember also the notation $[\mathbf{p}(x, z, \omega), \omega) = (z_\omega(L), z_\omega(L+1), \ldots)$ for $L = \mathrm{d}(z, \mathbf{p}(x, z, \omega))$. If above composition is not reduced, then $z_\omega(L + 1) = x_{k-1}$ and one finds $\mathrm{d}(x, z_\omega(L + 1)) = k - 1 < \mathrm{d}(x, \mathbf{p}(x, z, \omega)) = \min_{z' \in [z,\omega)} \mathrm{d}(x, z')$ in contradiction. This shows

$$\mathbf{p}(x, z, \omega) \in [x, \omega). \tag{4.5}$$

With equation (4.5) it is clear that $\mathrm{d}(z, \mathbf{p}(z, x, \omega)) = \min_{x' \in [x,\omega)} \mathrm{d}(z, x') \leq \mathrm{d}(z, \mathbf{p}(x, z, \omega))$ and similarly $\mathrm{d}(x, \mathbf{p}(x, z, \omega)) \leq \mathrm{d}(x, \mathbf{p}(z, x, \omega))$, whence the two rays

$$\begin{gathered} [\mathbf{p}(x, z, \omega), \mathbf{p}(z, x, \omega)][\mathbf{p}(z, x, \omega), \omega) \\ [\mathbf{p}(z, x, \omega), \mathbf{p}(x, z, \omega)][\mathbf{p}(x, z, \omega), \omega) \end{gathered}$$

are reduced. By uniqueness of reduced paths, we can substitute the lower one into the top one and obtain that $[\mathbf{p}(x, z, \omega), \mathbf{p}(z, x, \omega)][\mathbf{p}(z, x, \omega), \mathbf{p}(x, z, \omega)]$ is a closed reduced path, hence has length zero. This shows

$$\mathbf{p}(z, x, \omega) = \mathbf{p}(x, z, \omega). \tag{4.6}$$

4.11 Definition (Intersection). The *intersection* of $[x, \omega)$ and $[z, \omega)$ is defined as the ray $[x, \omega) \cap [z, \omega) := [t, \omega)$ for $t = \mathbf{p}(x, z, \omega)$ (see Figure 4.4). Since

Figure 4.4: Intersection of rays leading to the same border point

the projection is symmetric in the vertices, we obtain commutativity of the intersection: $[x,\omega) \cap [z,\omega) = [z,\omega) \cap [x,\omega)$.

4.12 Definition (Horocycles). For vertices x, y and a border point ω we introduce the relation

$$x \sim_\omega z :\Longleftrightarrow \mathrm{d}(x, \mathbf{p}(x,z,\omega)) = \mathrm{d}(z, \mathbf{p}(z,x,\omega)) \tag{4.7}$$

We verify below, that $\sim_\omega$ is an equivalence relation on $\mathrm{V}\mathcal{T}$ for all $\omega \in \mathcal{T}(\infty)$. The classes are called *ω-horocycles*. We will omit the index ω to $\sim$ if there is no danger of confusion. The integer number $B_\omega(x,z) := \mathrm{d}(x, \mathbf{p}(x,z,\omega)) - \mathrm{d}(z, \mathbf{p}(z,x,\omega))$ is called the *ω-horocycle distance from x to z*.

A direct result of (4.6) is $B_\omega(x,z) = -B_\omega(z,x)$ for all arguments $x, y \in \mathrm{V}\mathcal{T}$, $\omega \in \mathcal{T}(\infty)$, in particular $B_\omega(x,x) = 0$.

4.13 Lemma. *Suppose $n \in \mathbb{Z}$, $x, y \in \mathrm{V}\mathcal{T}$ and $\omega \in \mathcal{T}(\infty)$. Then $B_\omega(x,z) = n$ if and only if $x_\omega(l+n) = z_\omega(l)$ for some $l \in \mathbb{N}_0$.*

Proof. If $B_\omega(x,z) = n \in \mathbb{Z}$, we can put $l := \mathrm{d}(z, \mathbf{p}(z,x,\omega)) = \mathrm{d}(x, \mathbf{p}(x,z,\omega)) - n$. Then

$$\begin{aligned} x_\omega(l+n) \quad &= \quad x_\omega(\mathrm{d}(x, \mathbf{p}(x,z,\omega))) \quad = \quad \mathbf{p}(x,z,\omega) \quad = \quad \mathbf{p}(z,x,\omega) \\ &= \quad z_\omega(\mathrm{d}(z, \mathbf{p}(z,x,\omega))) \quad = \quad z_\omega(l). \end{aligned}$$

The opposite is also true. If $x_\omega(l+n) = z_\omega(l)$ for some $l \in \mathbb{N}_0$, then

$$\mathrm{d}(x, \mathbf{p}(x,z,\omega)) = \min_{z' \in \mathrm{V}z_\omega} \mathrm{d}(x,z') \leq \mathrm{d}(x, z_\omega(l)) = \mathrm{d}(x, x_\omega(l+n)).$$

By equation (4.5), the projection $\mathbf{p}(x,z,\omega)$ is a vertex in $[x,\omega)$. With above inequality follows $\mathbf{p}(x,z,\omega \in [x, x_\omega(l+n)]$ so that Lemma 1.25 shows

$$\mathrm{d}(x, x_\omega(l+n)) = \mathrm{d}(x, \mathbf{p}(x,z,\omega)) + \mathrm{d}(\mathbf{p}(x,z,\omega), x_\omega(l+n)).$$

An analogous equation can be proved in the same way for z stating $\mathrm{d}(z, z_\omega(l)) = \mathrm{d}(z, \mathbf{p}(z, x, \omega)) + \mathrm{d}(\mathbf{p}(z, x, \omega), z_\omega(l))$. This allows to calculate

$$\begin{aligned} B_\omega(x, z) &= \mathrm{d}(x, \mathbf{p}(x, z, \omega)) - \mathrm{d}(z, \mathbf{p}(z, x, \omega)) \\ &= \mathrm{d}(x, x_\omega(l+n)) - \mathrm{d}(z, z_\omega(l)) = n, \end{aligned}$$

using $x_\omega(l + n) = z_\omega(l)$ when substituting above equations into $B_\omega(x, z)$. □

We have now the tools to prove, that $\sim_\omega$ is an equivalence relation. Reflexivity and symmetry follow from definition and symmetry of $\mathbf{p}(x, z, \omega)$ in the vertex variables. For transitivity we suppose $x \sim y$ and $y \sim z$ hence by the previous lemma we get $x_\omega(k) = y_\omega(k)$ and $y_\omega(l) = z_\omega(l)$ for some $k, l \in \mathbb{N}_0$. If $k \leq l$, the infinite intersection property gives $x_\omega(l) = y_\omega(l)$, hence $x_\omega(l) = z_\omega(l)$, whence by the previous lemma $x \sim z$. For $l \leq k$ we argue similarly.

4.14 Lemma. *The horocycle distance $B_\omega(\cdot, \cdot)$ is constant on ω-horocycles.*

Proof. Suppose that $B_\omega(x, z) = n$. Lemma 4.13 states, that for some $l \in \mathbb{N}_0$ one has $x_\omega(l + n) = z_\omega(l)$. If a vertex y satisfies $y \sim_\omega z$ then $B_\omega(y, z) = 0$, hence for some $k \in \mathbb{N}_0$ we have $y_\omega(k) = z_\omega(k)$. For $L := \max\{k, l\}$ we get $x_\omega(L + n) = z_\omega(L) + y_\omega(L)$ and therefore $B_\omega(x, y) = n = B_\omega(x, z)$. This shows the assertion for the second variable of B_ω. The same is true for the first variable of B_ω, since $B_\omega(y, x) = -B_\omega(x, y) = -B_\omega(x, z) = B_\omega(z, x)$. □

4.15 Lemma. *Suppose, r is a bi-infinite reduced path and a pair of vertices $r(k), r(l)$ is given. Then*

$$\begin{aligned} (r(0), r(1), r(2), \ldots) \in \omega &\implies B_\omega(r(k), r(l)) = l - k \\ (r(0), r(-1), r(-2), \ldots) \in \omega &\implies B_\omega(r(k), r(l)) = k - l. \end{aligned}$$

Proof. By definition of the horocycle distance

$$B_\omega(r(k), r(l)) = \mathrm{d}(r(k), \mathbf{p}(r(k), r(l), \omega)) - \mathrm{d}(r(l), \mathbf{p}(r(l), r(k), \omega)).$$

If $k \leq l$ then $\mathbf{p}(r(k), r(l), \omega) = r(l)$, hence $B_\omega(r(k), r(l)) = \mathrm{d}(r(k), r(l)) = |l - k| = l - k$. If $l \leq k$ then $\mathbf{p}(r(l), r(k), \omega) = r(k)$, hence $B_\omega(r(k), r(l)) = -\mathrm{d}(r(k), r(l)) = -|k - l| = l - k$. For the second assertion, consider the path defined by $s(i) := r(-i)$ for all $i \in \mathbb{Z}$, which clearly satisfies the condition of the first assertion, therefore $B_\omega(r(k), r(l)) = B_\omega(s(-k), s(-l)) = (-l) - (-k) = k - l$. □

In the notation of a ray by a vertex and a border point, the above Lemma can be expressed as follows.

4.16 Corollary. *For each* $\omega \in \mathcal{T}(\infty)$ *and* $x \in \mathrm{V}\mathcal{T}$ *holds* $B_\omega(x_\omega(k), x_\omega(l)) = l - k$ *for all* $l, k \in \mathbb{N}_0$.

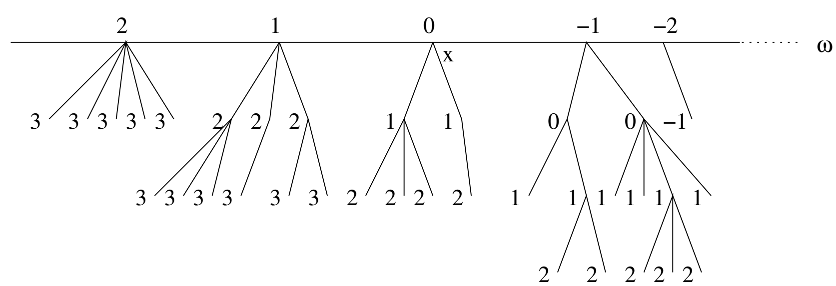
4.17 Lemma. *A reduced bi-infinite path* $r = \ldots, r(-1), r(0), r(1), \ldots$ *with* $r(0), r(1), r(2), \ldots \in \omega$ *intersects each* ω*-horocycle exactly in one vertex.*

Proof. If $z \in H$ for some ω-horocycle H, and $n = B_\omega(r(0), z)$ we set $r_n := [r(n), \omega)$. Lemma 4.13 gives $r_n(l) = r(l+n) = z_\omega(l)$ for some $l \in \mathbb{N}_0$, whence $B_\omega(r(n), z) = 0$ gives $r(n) \sim_\omega z$. This shows $r(n) \in H$ because ω-horocycles are the classes of the equivalence relation $\sim_\omega$, and we can deduce, that each ω-horocycle contains at least one vertex of r.

If there are two vertices $r(l), r(k)$ in the same horocycle, the proof is completed by $k - l = B_\omega(r(l), r(k)) = B_\omega(r(l), r(l)) = 0$ through Lemma 4.14 and Lemma 4.15. □

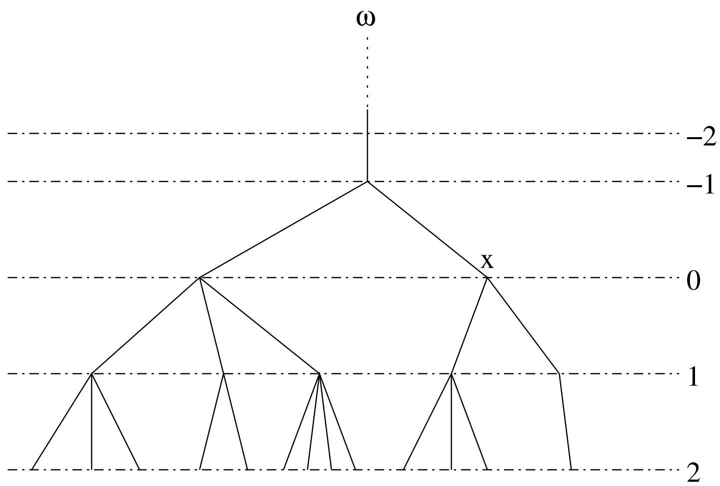
These results enable us to label the ω-horocycles of $\mathrm{V}\mathcal{T}$ by integer numbers. Confer Figure 4.5 and Figure 4.6.

Figure 4.5: Horocycles: Vertices are accompanied by the horocycle distance $B_\omega(\cdot, x)$ to the vertex x.



Given three vertices x, y, z of $\mathcal{T}$ and a border point $\omega \in \mathcal{T}(\infty)$, we choose any reduced bi-infinite path $r \in \mathcal{RT}$ such that $r(0), r(1), r(2), \ldots \in \omega$. By the previous lemma there are numbers $k, l, m \in \mathbb{Z}$, such that $r(k) \sim x$, $r(l) \sim y$ and

Figure 4.6: More Horocycles. (Meaning of numbers as in Figure 4.5)



$r(m) \sim z$, hence

$$\begin{aligned} B_\omega(x,y) + B_\omega(y,z) &= B_\omega(r(k), r(l)) + B_\omega(r(l), r(m)) \\ = \quad l - k + m - l &= m - k \\ = \quad B_\omega(r(k), r(m)) &= B_\omega(x,z). \end{aligned}$$

As a summary we can write for all $x, y, z \in \mathrm{V}\mathcal{T}$ and all $\omega \in \mathcal{T}(\infty)$

$$\begin{aligned} B_\omega(x,x) &= 0 \\ B_\omega(x,y) &= -B_\omega(y,x) \\ B_\omega(x,y) + B_\omega(y,z) &= B_\omega(x,z). \end{aligned} \tag{4.8}$$

The section is abandoned with an equation for the horocycle distance involving isometric actions on its variables:

4.18 Lemma. *Given an isometry $h \in Is(\mathcal{T})$, $x, y \in V\mathcal{T}$ and $\omega \in \mathcal{T}(\infty)$, one has $d(x, \mathbf{p}(x,y,\omega)) = d(hx, \mathbf{p}(hx,hy,h\omega))$.*

Proof.

$$\begin{aligned} \mathrm{d}(hx, \mathbf{p}(hx,hy,h\omega)) &= \min_{z \in [hy, h\omega)} \mathrm{d}(hx, z) \\ = \min_{hz' \in [hy,h\omega)} \mathrm{d}(hx, hz') &= \min_{z' \in [y,\omega)} \mathrm{d}(x, z') \\ = \mathrm{d}(x, \mathbf{p}(x,y,\omega)). \end{aligned}$$

□

Directly from definition follows with this argument

$$B_{h\omega}(hx, hz) = B_\omega(x, z) \tag{4.9}$$

for all $h \in \mathrm{Is}\,(\mathcal{T})$, $x, z \in \mathrm{V}\mathcal{T}$ and $\omega \in \mathcal{T}(\infty)$. Statements like this one are often considered as *transport of structure* in the literature.

4.4 Topology on the border of a tree

A metric on a set can be used to define a topology for this set, referred to as *metric topology* (cf. [18]). The definition is based on the declaration of the family of open spheres of the metric as the basis for a topology. In the reference above is a proof that this method can be applied. In our case, the metric $\mathrm{d}_x(\cdot,\cdot)$ assumes only the values

$$1, e^{-1}, e^{-2}, e^{-3}, \dots \quad \text{and} \quad 0.$$

A sphere of radius zero is empty and each subsequent pair $e^{-k}, e^{-(k+1)}$ has a positive distance. Therefore the open spheres about a border point $\omega \in \mathcal{T}(\infty)$ with respect to the metric d_x can be labeled for $n \in \mathbb{N}_0$ as

$$\Omega_n^x(\omega) := \{\eta \in \mathcal{T}(\infty) : \mathrm{d}_x(\omega, \eta) \le e^{-n}\}$$

or equivalently as

$$\Omega_n^x(\omega) = \{\eta \in \mathcal{T}(\infty) : L_{\omega\eta}^x \ge n\}.$$

In particular $\Omega_0^x(\omega) = \mathcal{T}(\infty)$. Figure 4.7 shows an illustration of $\Omega_3^x(\omega)$ represented by reduced rays in the relative border $\mathcal{T}_x(\infty)$ with common origin x.

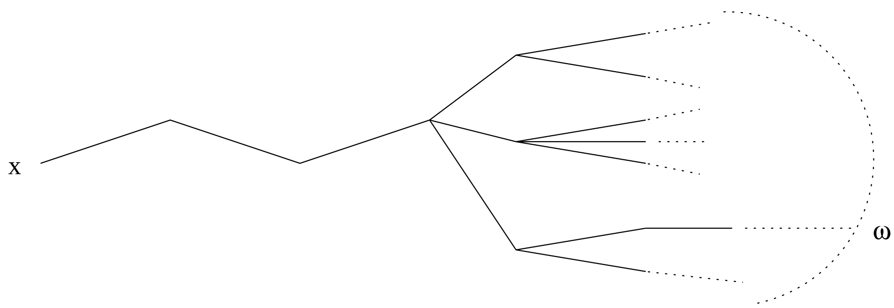
It is useful to have an additional way of writing the open spheres $\Omega_n^x(\omega)$. Therefore we introduce for $x, y \in \mathrm{V}\mathcal{T}$ the set

$$\Omega(x, y) := \{\omega \in \mathcal{T}(\infty) : y \in [x, \omega)\}\,.$$

In the representation by elements of $\mathcal{T}_x(\infty)$, this set consists of all rays starting at x and passing through y. Every sphere in $\mathcal{T}(\infty)$ can be written in the form $\Omega(x, y)$ because

$$\Omega_n^x(\omega) = \Omega(x, x_\omega(n)). \tag{4.10}$$

Figure 4.7: An open sphere of radius e^{-3}



In more detail this can be shown step by step as

$$\begin{aligned}\eta \in \Omega_n^x(\omega) &\Leftrightarrow L_{\eta\omega}^x \geq n \\ &\Leftrightarrow [x,\eta) = [x, x_\omega(n)][x_\omega(n), \eta) \\ &\Leftrightarrow x_\omega(n) \in [x,\eta) \\ &\Leftrightarrow \eta \in \Omega(x, x_\omega(n)).\end{aligned}$$

Equation (4.10) proves, that the family of open spheres to the metric d_x can be written as $\{\Omega(x,y)\}_{y\in \mathrm{V}\mathcal{T}}$. By Theorem 5 in [18] a metric topology satisfies the *second countability axiom*, i.e. there is a countable basis for this topology. Indeed we know now that the basis $\{\Omega_n^x(\omega)\}_{n\in\mathbb{N},\omega\in\mathcal{T}(\infty)}$ is countable itself since $\mathrm{V}\mathcal{T}$ is countable as $\mathcal{T}$ is a locally finite tree. So the most general open set is a countable union of these spheres.

We give some more technical details before making convenient statements: For each two vertices $x, y \in \mathrm{V}\mathcal{T}$ is

$$\begin{aligned}\Omega(x,y) &= \{\eta \in \mathcal{T}(\infty) : y \in [x,\eta)\} \\ &= \{\eta \in \mathcal{T}(\infty) : z \in [x,\eta), \quad y \in [x,z]\} \\ &\overset{Lemma\ 1.25}{=} \{\eta \in \mathcal{T}(\infty) : z \in [x,\eta), \quad [x,y][y,z] \text{ is reduced }\} \\ &= \bigcup_{\substack{[x,y][y,z] \\ \text{is reduced}}} \Omega(x,z).\end{aligned} \tag{4.11}$$

In particular if $[x,y][y,z]$ is reduced, then

$$\Omega(x,z) \subset \Omega(x,y). \tag{4.12}$$

If we suppose that $\mathrm{len}[x,y] = m \geq 0$ and $\mathrm{len}[y,z] = n \geq 0$ then $\eta \in \Omega(x,z) \Leftrightarrow x_\eta(m+n) = z$. Hence the union $\bigcup_{\substack{[x,y][y,z] \text{ is reduced} \\ \mathrm{len}[y,z]=n}} \Omega(x,z)$ is disjoint. Also each reduced ray from x passing through y has a vertex in distance $m+n$ from x, thus for all $n \in \mathbb{N}_0$

$$\Omega(x,y) = \bigsqcup_{\substack{[x,y][y,z] \text{ is reduced} \\ \operatorname{len}[y,z]=n}} \Omega(x,z). \tag{4.13}$$

If we apply equation (4.13) to the case $x = y$, we obtain for all $n \in \mathbb{N}_0$

$$\mathcal{T}(\infty) = \bigsqcup_{\substack{[x,z] \text{ is reduced} \\ \operatorname{len}[x,z]=n}} \Omega(x,z) = \bigsqcup_{\mathrm{d}(x,z)=n} \Omega(x,z).$$

This shows that $\Omega(x,z) = \mathcal{T}(\infty) \setminus \bigsqcup_{\substack{\mathrm{d}(x,z')=\mathrm{d}(x,z) \\ z' \neq z}} \Omega(x,z')$, whence all open spheres are also closed.

Since $(\mathcal{T}_x(\infty), \mathrm{d}_x)$ is a metric space and each sequence has a convergent subsequence (cf. Section 4.2), Theorem 5 in [18] proves that $\mathcal{T}(\infty)$ is compact. In particular the open spheres are compact as closed subsets of a compact space. We give a summary of what has been said about topology of the border so far:

4.19 Proposition. *The base* $\{\Omega_n^x(\omega)\}_{n \in \mathbb{N}_0, \omega \in \mathcal{T}(\infty)}$ *of the metric topology of* d_x *equals* $\{\Omega(x,z)\}_{z \in \mathrm{V}\mathcal{T}}$. *All members of this countable family are open, closed and compact. In particular* $\mathcal{T}(\infty)$ *is compact.*

There is more deduction from metric properties of the visual metrics for metric the topologies.

4.20 Proposition. *Each metric* d_x *on* $\mathcal{T}(\infty)$ *induces the same metric topology on* $\mathcal{T}(\infty)$.

Proof. Proposition 4.7 states that two metrics are equivalent in the sense that

$$e^{-\mathrm{d}(x,y)} \mathrm{d}_x(\eta,\xi) \leq \mathrm{d}_y(\eta,\xi) \leq e^{\mathrm{d}(x,y)} \mathrm{d}_x(\eta,\xi)$$

for all $x, y \in \mathrm{V}\mathcal{T}$ and $\eta, \xi \in \mathcal{T}(\infty)$. This implies, that the metric topologies of d_x and d_y coincide (confer [18]). $\square$

This topology, which is the metric topology to any of the visual metrics d_x, will be referred to simply as *the topology of* $\mathcal{T}(\infty)$.

For use in the next section, we prepend one more useful technical statement:

$$\Omega(x,y) = \Omega(x_{n-1}, x_n) \tag{4.14}$$

for $[x,y] = x_0, \ldots, x_n$ and $n \geq 1$. This follows from

$$
\begin{aligned}
\eta \in \Omega(x,y) \quad &\Leftrightarrow \quad y \in [x,\eta) \quad \Leftrightarrow \quad [x_0,x_n][x_n,\eta) \text{ is reduced} \\
&\Leftrightarrow \quad [x_{n-1},x_n][x_n,\eta) \text{ is reduced} \quad \Leftrightarrow \quad x_n \in [x_{n-1},\eta) \\
&\Leftrightarrow \quad \eta \in \Omega(x_{n-1},x_n).
\end{aligned}
$$

A set $M \subset \mathcal{T}(\infty)$ is *connected*, if it is not a union of two non–empty sets $M \cap A$ and $M \cap B$ where A and B are both open and disjoint. If a set $M \in \mathcal{T}(\infty)$ contains two points $\eta \neq \xi$, then $\mathrm{d}_x(\eta,\xi)$ is positive. Hence $L^x_{\eta\xi} < \infty$ and for $n = L^x_{\eta\xi}+1$ we have $x_\eta(n) \neq x_\xi(n)$. As both $A = \Omega(x, x_\eta(n))$ and $B = \mathcal{T}(\infty)\setminus A$ are open, $\eta \in A$ and $\xi \in B$, the equation

$$
M = M \cap \mathcal{T}(\infty) = (M \cap A) \cup (M \cap B)
$$

shows that M is not connected. Only subsets of $\mathcal{T}(\infty)$ with one element are connected. Topological spaces with this property are called *totally disconnected*.

4.5 Locally constant functions

A function $F : \mathcal{T}(\infty) \to \mathbb{C}$ is called *locally constant*, if for every $\omega \in \mathcal{T}(\infty)$ there exists an open set M, such that $\omega \in M$ and F is constant on M, i.e. $F(\eta) = F(\omega)$ for all $\eta \in M$. We already know many examples of locally constant functions:

4.21 Lemma. *For fixed $x, y \in \mathrm{V}\mathcal{T}$ the function $\omega \mapsto B_\omega(x,y)$ is a locally constant function on $\mathcal{T}(\infty)$.*

Proof. For $x, y \in \mathrm{V}\mathcal{T}$ and $\omega \in \mathcal{T}(\infty)$ $[x,\omega)$ and $[y,\omega)$ have infinite intersection, say $x_\omega(n) = y_\omega(m)$. This implies by Lemma 4.13 $B_\omega(x,y) = n - m$. Observe also $y_\omega(m+1) = x_\omega(n+1)$. We want to show that $B_\omega(x,y)$ is constant on $\Omega^x_{n+1}(\omega)$. Say $\eta \in \Omega^x_{n+1}(\omega)$. This implies $x_\eta(n+1) = x_\omega(n+1)$. Furthermore

$$
\begin{aligned}
\Omega^x_{n+1}(\omega) \quad &= \quad \Omega(x, x_\omega(n+1)) \quad \overset{(4.14)}{=} \quad \Omega(x_\omega(n), x_\omega(n+1)) \\
&= \quad \Omega(y_\omega(m), y_\omega(m+1)) \quad \overset{(4.14)}{=} \quad \Omega(y, y_\omega(m+1)) \quad = \quad \Omega^y_{m+1}(\omega),
\end{aligned}
$$

whence $y_\eta(m+1) = y_\omega(m+1)$. Finally, by Lemma 4.13 holds $B_\eta(x,y) = (n+1) - (m+1) = B_\omega(x,y)$. ∎

It is important to observe, that the set

$$S(\mathcal{T}(\infty))$$

of all locally constant functions on $\mathcal{T}(\infty)$ is a vector space. This is due to the fact that the set of maps from some set to a vector space is a vector space and that the intersection of two open sets is an open set. The action of $\mathrm{Is}(\mathcal{T})$ on $\mathcal{T}(\infty)$ induces an action on $S(\mathcal{T}(\infty))$ by

$$h * \varphi(\eta) := \varphi(h^{-1}\eta) \tag{4.15}$$

for $h \in \mathrm{Is}(\mathcal{T})$, $\varphi \in S(\mathcal{T}(\infty))$ and $\eta \in \mathcal{T}(\infty)$. Firstly it is a general property, that a group acting on some set acts also on every function space over this set. This can be written as $(\mathrm{Id} * \varphi)(\eta) = \varphi(\mathrm{Id}^{-1}\eta)$ and $h * (g * \varphi)(\eta) = g * \varphi(h^{-1}\eta)$ $= \varphi(g^{-1}h^{-1}\eta) = \varphi((hg)^{-1}\eta) = (hg) * \varphi(\eta)$ for all $g, h \in \mathrm{Is}(\mathcal{T})$, $\varphi \in S(\mathcal{T}(\infty))$ and $\eta \in \mathcal{T}(\infty)$.

The fact that $h \in \mathrm{Is}(\mathcal{T})$ indeed maps locally constant functions to locally constant function is due to the property of isometries mapping continuously from $\mathcal{T}(\infty)$ to $\mathcal{T}(\infty)$: By equation (4.3) holds $\mathrm{d}_x(\eta, \xi) = \mathrm{d}_{hx}(h\eta, h\xi)$ $x \in \mathrm{V}\mathcal{T}$, $\eta, \xi \in \mathcal{T}(\infty)$. Hence

$$\begin{aligned} h^{-1}\Omega_n^x(\omega) \quad &= \quad \{h^{-1}\eta \in \mathcal{T}(\infty) : \mathrm{d}_x(\omega, \eta) \le e^{-n}\} \\ &= \quad \{\xi \in \mathcal{T}(\infty) : \mathrm{d}_x(\omega, h\xi) \le e^{-n}\} \\ &= \quad \{\xi \in \mathcal{T}(\infty) : \mathrm{d}_{h^{-1}x}(h^{-1}\omega, \xi) \le e^{-n}\} \quad = \quad \Omega_n^{h^{-1}x}(h^{-1}\omega). \end{aligned}$$

Given $\omega \in \mathcal{T}(\infty)$, $\varphi \in S(\mathcal{T}(\infty))$ is constant on some $\mathrm{d}_{h^{-1}x}$-sphere about $h^{-1}\omega$, by Proposition 4.19 and Proposition 4.20. Say φ is constant on $\Omega_n^{h^{-1}x}(h^{-1}\omega)$. Then $h * \varphi(\omega) = \varphi(h^{-1}\omega)$ shows with above equation that $h * \varphi$ is constant on the d_x-sphere $\Omega_n^x(\omega)$ about ω.

For later use we write the action of isometries on spheres of the form $\Omega(x, y)$. Assume $\mathrm{d}(x, y) = n \ge 0$, $\eta \in \Omega(x, y)$ and $h \in \mathrm{Is}(\mathcal{T})$. Then $h\Omega(x, y) = h\Omega_n^x(\eta) =$ $\Omega_n^{hx}(h\eta)$ by above equation. Since $(hx)_{(h\eta)}(n) \overset{(4.2)}{=} h(x_\eta)(n) \overset{(4.1)}{=} h(x_\eta(n)) = hy$ one has

$$h\Omega(x, y) = \Omega(hx, hy) \tag{4.16}$$

for all $x, y \in \mathrm{V}\mathcal{T}$ and all $h \in \mathrm{Is}(\mathcal{T})$.

We introduce for sets $M \subset \mathcal{T}(\infty)$ the *characteristic function*

$$\chi_M : \begin{array}{ccc} \mathcal{T}(\infty) & \longrightarrow & \mathbb{C} \\ \omega & \longmapsto & \begin{cases} 1 & \text{if } \omega \in M \\ 0 & \text{if } \omega \notin M \end{cases} \end{array} .$$

By definition we can find for every locally constant function $\varphi \in S(\mathcal{T}(\infty))$ a family $\{O_\alpha\}$ of open sets such that φ is constant on each member. Choosing a vertex $x \in \mathrm{V}\mathcal{T}$, we can use the open spheres of d_x to write each member of $\{O_\alpha\}$ as a union of $\Omega(x,z)$'s and consider the cover of these spheres instead. By compactness of $\mathcal{T}(\infty)$ this new cover has a finite subcover and we can write

$$\varphi = \alpha(z_1)\chi_{\Omega(x,z_1)} + \cdots + \alpha(z_k)\chi_{\Omega(x,z_k)} \tag{4.17}$$

for some $\alpha(z_i) \in \mathbb{C}$, $z_i \in \mathrm{V}\mathcal{T}$ $(1 \le i \le k)$ and $k \in \mathbb{N}$. Such a decomposition can be done for every locally constant function, hence $S(\mathcal{T}(\infty))$ is generated by $\{\chi_{\Omega(x,z)}\}_{z \in \mathrm{V}\mathcal{T}}$ and we can write

$$S(\mathcal{T}(\infty)) = \left\{ \sum_{z \in V'} \alpha(z)\chi_{\Omega(x,z)} : \alpha(z) \in \mathbb{C}, V' \subset \mathrm{V}\mathcal{T} \text{ finite} \right\}. \tag{4.18}$$

Note that each of the sets $\Omega(x, z_i)$ appearing in equation (4.17) can be split up by equation (4.13) into a disjoint union

$$\Omega(x, z_i) = \bigsqcup_{\substack{[x,z_i][z_i,z'] \text{ is reduced} \\ \mathrm{d}(x,z')=n}} \Omega(x, z')$$

whenever $n \ge N := \max_{1 \le i \le k} \mathrm{d}(x, z_i)$. Hence for each $\varphi \in S(\mathcal{T}(\infty))$ there is $N \in \mathbb{N}_0$ such that for all $n \ge N$

$$\varphi = \sum_{\mathrm{d}(x,z)=n} \beta(z)\chi_{\Omega(x,z)} \tag{4.19}$$

for some constants $\beta(z) \in \mathbb{C}$. We can write a modified version of equation (4.18) as

$$S(\mathcal{T}(\infty)) = \left\{ \sum_{\mathrm{d}(x,z)=n} \beta(z)\chi_{\Omega(x,z)} : \beta(z) \in \mathbb{C}, n \in \mathbb{N}_0, n \ge K \right\} \tag{4.20}$$

for all $K \in \mathbb{N}_0$.

Part III

Application

Chapter 5

A dynamical system

5.1 Setup and overview

The dynamical system we are going to work with is based on the *geodesic space* $\mathcal{G}(A, \mathrm{i}_A)$ of a finite connected edge-indexed graph (A, i_A). Elements of the geodesic space are bi-infinite geodesics in (A, i_A), we call them *geodesics* (confer Chapter 2).

On top of the geodesic space we put a shift operator L, which acts as a left shift on each geodesic g:

$$(Lg)\,[i, i+1] := g[i+1, i+2] \quad \text{for all} \quad i \in \mathbb{Z}. \tag{5.1}$$

L maps geodesics to geodesics, since its action can be regarded as a right shift R on the domain of definition $\mathcal{T}_2$ of a geodesic g. Explicitly, $R(i) := i+1$ defines an automorphism of $\mathcal{T}_2$ (cf. 1.23) such that $R[i, i+1] = [i+1, i+2]$. Hence gR is a geodesic which equals Lg since $gR[i, i+1] = g[i+1, i+2] = Lg[i, i+1]$ for all $i \in \mathbb{Z}$.

The operator L^{-1} defined by $(L^{-1}g)[i, i+1] := g[i-1, i]$ is left and right inverse to L. Hence the shift operator L is invertible and generates a group. We set $L(n) := L^n$ for all $n \in \mathbb{Z}$ to obtain a group homomorphism from the *time* $\mathbb{Z}$ to the set of invertible transformations from $\mathcal{G}(A, \mathrm{i}_A)$ to $\mathcal{G}(A, \mathrm{i}_A)$. In other words, the time $\mathbb{Z}$ acts on the geodesic space $\mathcal{G}(A, \mathrm{i}_A)$. The geodesic space together with the shift operator, $(\mathcal{G}(A, \mathrm{i}_A), L)$, is called a *dynamical system*. We are going to follow two main lines for analyzing this dynamical system (see

Figure 5.1: Translations of the geodesic space

$$
\begin{array}{ccccc}
 & \diagup & \mathcal{R}(\mathcal{T}) & \text{———} & \mathcal{T}(\infty) \times \mathcal{T}(\infty) \times \mathbb{Z} \\
\mathcal{G}(A, \mathrm{i}_A) & & & & \\
 & \diagdown & \mathcal{P}(\mathcal{L}^+(A, \mathrm{i}_A)) & &
\end{array}
$$

Figure 5.1).

One easy to understand yet very clarifying correspondence is that one of $\mathcal{G}(A, \mathrm{i}_A)$ with the space $\mathcal{P}(\mathcal{L}^+(A, \mathrm{i}_A))$ of positive bi-infinite paths in the oriented line graph $\mathcal{L}^+(A, \mathrm{i}_A))$ (cf. Section 2.2). The assignment $(\mathrm{C}_\mathcal{P} g)(i) := g[i, i+1]$ for all $i \in \mathbb{Z}$ defines a bijection $\mathrm{C}_\mathcal{P} : \mathcal{G}(A, \mathrm{i}_A) \to \mathcal{P}(\mathcal{L}^+(A, \mathrm{i}_A))$. We can define an operator $L_\mathcal{P}$ on $\mathcal{P}(\mathcal{L}^+(A, \mathrm{i}_A))$ by $L_\mathcal{P} := \mathrm{C}_\mathcal{P} L \mathrm{C}_\mathcal{P}^{-1}$ to make the diagram

$$
\begin{array}{ccc}
\mathcal{G}(A, \mathrm{i}_A) & \xrightarrow{L} & \mathcal{G}(A, \mathrm{i}_A) \\
{\scriptstyle \mathrm{C}_\mathcal{P}} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{C}_\mathcal{P}} \\
\mathcal{P}(\mathcal{L}^+(A, \mathrm{i}_A)) & \xrightarrow[L_\mathcal{P}]{} & \mathcal{P}(\mathcal{L}^+(A, \mathrm{i}_A))
\end{array}
$$

commute, i.e. $L_\mathcal{P} \circ \mathrm{C}_\mathcal{P} = \mathrm{C}_\mathcal{P} \circ L$. Now the time $\mathbb{Z}$ acts via $L_\mathcal{P}$ on $\mathcal{P}(\mathcal{L}^+(A, \mathrm{i}_A))$ and the $\mathbb{Z}$-action of the time commutes with the identification $\mathrm{C}_\mathcal{P}$. A direct calculation yields for any positive path $p \in \mathcal{P}(\mathcal{L}^+(A, \mathrm{i}_A))$

$$
\begin{aligned}
(L_\mathcal{P} p)(i) \quad &= \quad (\mathrm{C}_\mathcal{P} L \mathrm{C}_\mathcal{P}^{-1} p)(i) \quad = \quad (L \mathrm{C}_\mathcal{P}^{-1} p)[i, i+1] \\
= \quad (\mathrm{C}_\mathcal{P}^{-1} p)[i+1, i+2] \quad &= \quad p(i+1).
\end{aligned}
$$

The action of $L_\mathcal{P}$ on a p corresponds therefore to a left shift, so the dynamical system $(\mathcal{P}(\mathcal{L}^+(A, \mathrm{i}_A)), L_\mathcal{P})$ corresponds to a *topological Markov chain* with *Markov graph* $\mathcal{L}^+(A, \mathrm{i}_A)$ and *Markov matrix* $[M]$ given by

$$
M_{a,b} = \begin{cases} 1 & \text{if } a, b \text{ is a geodesic in } (A, \mathrm{i}_A) \\ 0 & \text{if } a, b \text{ is not a geodesic in } (A, \mathrm{i}_A) \end{cases} \tag{5.2}
$$

for all vertices a and b of $\mathcal{L}^+(A, \mathrm{i}_A)$. (Confer [3], page 50.)

A survey of this chapter follows. For the main part attention shall be drawn to the top line of Figure 5.1. Fixing a base point $x_0 \in \mathrm{V}A$ there is a tree $\mathcal{T} = (\widetilde{A, \mathrm{i}_A}, x_0)$ called the universal cover of (A, i_A) and a group $G = \pi_1(A, \mathrm{i}_A, x_0)$

called the fundamental group, such that G acts on $\mathcal{T}$ with quotient graph $A = G\backslash\mathcal{T}$ and quotient morphism $\pi : \mathcal{T} \to A$ (cf. Chapter 3). An identification of the geodesic space $\mathcal{G}(A, \mathrm{i}_A)$ with a quotient of $\mathcal{R}(\mathcal{T})$ by left action of the full group G_f is written in Section 5.2. Topological arguments about the group G_f are used. Coordinates are introduced in Section 5.3 for reduced paths of the cover $\mathcal{R}\mathcal{T}$. A path is expressed in terms of two distinct border points and an integer value. The action of the time shift and the action of isometries are written in these coordinates. In Section 5.4 we relate α-dimensional densities on the border of $\mathcal{T}$ to G_f-invariant eigenfunctions F of a linear operator R on the function space over the edge set of $\mathcal{T}$. The central isomorphism of vector spaces will be proved in detail. Section 5.5 deals with questions of existence of that eigenfunctions. By G_f-invariance of R, the operator can be translated to the finite edge set $\mathrm{E}A$. Its coordinates will be calculated and eigenfunctions can be identified through the Perron Frobenius Theorem. The functions in turn will be used in Section 5.6 to write invariant measures for the topological Markov chain $(\mathcal{P}(\mathcal{L}^+(A, \mathrm{i}_A)), L_\mathcal{P})$. Indeed they define Markov measures, if the full group G_f is uni-modular. They have a time-reversal symmetry. In the special setting of graphs with minimal edge indexing one obtains Perry measures. Section 5.7 discusses the ergodic properties of the topological Markov chain $(\mathcal{P}(\mathcal{L}^+(A, \mathrm{i}_A)), L_\mathcal{P})$ endowed with the measures of Section 5.6 (and more general Markov Measures) following well known standards of symbolic dynamics. We are however in the position to relate these properties directly to the edge-indexed graph (A, i_A) through the results of Sections 2.4 and 2.5.

5.2 Translation to the universal cover

The results of this section allow us to consider a dynamical system on $\mathcal{T}$ rather than $(\mathcal{G}(A, \mathrm{i}_A), L)$ itself.

There is a shift operator $L_\mathcal{R}$ on $\mathcal{R}\mathcal{T}$, which preserves the classes of the action of G_f. For a reduced path r we define $L_\mathcal{R}$ by $(L_\mathcal{R} r)[i, i+1] := r[i+1, i+2]$ for $i \in \mathbb{Z}$. $L_\mathcal{R}$ acts on $\mathcal{R}(\mathcal{T})$ by the same arguments as L acts on $\mathcal{G}(A, \mathrm{i}_A)$. The diagram

$$\begin{array}{ccc} \mathcal{RT} & \xrightarrow{h\in \mathrm{Is}(\mathcal{T})} & \mathcal{RT} \\ {\scriptstyle L_{\mathcal{R}}}\big\downarrow & & \big\downarrow{\scriptstyle L_{\mathcal{R}}} \\ \mathcal{RT} & \xrightarrow[h]{} & \mathcal{RT} \end{array}$$

commutes, since

$$\begin{aligned} L_{\mathcal{R}} \circ h(r)[i, i+1] \quad &= \quad h(r)[i+1, i+2] \quad = \quad h(r[i+1, i+2]) \\ = \quad h((L_{\mathcal{R}}r)[i, i+1]) \quad &= \quad h \circ L_{\mathcal{R}}(r)[i, i+1] \end{aligned}$$

for all $i \in \mathbb{Z}$ and all reduced paths r. Fixing a reduced path r we can now write $L_{\mathcal{R}}h(r) = hL_{\mathcal{R}}(r)$. Taking the union over all isometries of G_f, this equation becomes

$$L_{\mathcal{R}}\,(G_f p) = G_f\,(L_{\mathcal{R}}p) \tag{5.3}$$

and shows, that $L_{\mathcal{R}}$ induces an operator acting on the space $G_f\backslash\mathcal{RT}$. We call this operator also $L_{\mathcal{R}}$. $L_{\mathcal{R}}$ defines now a $\mathbb{Z}$-action on $G_f\backslash\mathcal{RT}$. We would like write a $\mathbb{Z}$-equivariant identification

$$G_f\backslash\mathcal{R}(\mathcal{T}) \longrightarrow \mathcal{G}(A, \mathrm{i}_A). \tag{5.4}$$

The crucial point will be to establish injectivity. An argument using compactness of vertex stabilizers will help.

5.1 Lemma. *The quotient morphism $\pi : \mathcal{T} \to A$ induces a map $\mathcal{R}(\mathcal{T}) \longrightarrow \mathcal{G}(A, \mathrm{i}_A)$ and a map $\pi : G_f\backslash\mathcal{R}(\mathcal{T}) \longrightarrow \mathcal{G}(A, \mathrm{i}_A)$.*

Proof. If we show that π maps reduced paths of length two to geodesics, this property then generalizes to bi-infinite reduced paths. Assume that (a, b) is a reduced path of length two, $x = \mathrm{o}(b)$. If $\pi(a, b) = (\pi(a), \pi(b))$ is not a geodesic, then $\pi(b) = \overline{\pi(a)}$ and $\mathrm{i}(\pi b) = 1$. By equation (3.8) one has $\pi_x^{-1}(\pi b) = \{b\}$ which equals $\pi_x^{-1}(\overline{\pi a}) = \{\overline{a}\}$. Hence $b = \overline{a}$, which is impossible for a reduced path. The second assertion becomes clear because π is G_f-invariant. □

5.2 Lemma. *The map $\pi : G_f\backslash\mathcal{R}(\mathcal{T}) \longrightarrow \mathcal{G}(A, \mathrm{i}_A)$ induced from the quotient morphism π of G_f is onto.*

Proof. Note that for any geodesic $(\mathbf{a}, \mathbf{b})$ of length two in (A, i_A) and any edge $a \in \pi^{-1}(\mathbf{a})$ there is an edge $b \in \pi^{-1}_{\mathrm{t}(a)}(\mathbf{b})$, such that a, b is reduced. Otherwise $\overline{a}$

is the only edge of $\mathrm{St}^{\mathcal{T}}(\mathrm{t}(a))$ projecting to $\mathbf{b}$ under π, hence by equation (3.8) $\mathrm{i}(\mathbf{b}) = 1$. Then $\mathbf{b} = \pi(\overline{a}) = \overline{\mathbf{a}}$ shows that $\mathbf{a}, \mathbf{b}$ was not a geodesic.

Given any geodesic g in $\mathcal{G}(A, \mathrm{i}_A)$ we can use Lemma 1.29 and the above remark to find inductively a reduced ray $c = c[0,1], c[1,2], \ldots$ in $\mathcal{T}$ such that

$$\pi(c) = g[0,1], g[1,2], g[2,3], \ldots .$$

Analogously we can construct a reduced ray $d = d[0,1], d[1,2], \ldots$ with $\pi(d) = \overline{g[0,1]}, \overline{g[-1,0]}, \overline{g[-2,-1]}, \ldots$ and $d[0,1] = \overline{c[0,1]}$. The path p with edge sequence

$$p[i, i+1] = \begin{cases} \overline{d[-i, -i+1]} & \text{for all} \quad i \leq 0 \\ c[i, i+1] & \text{for all} \quad i \geq 1 \end{cases}$$

is then a reduced bi-infinite path in $\mathcal{T}$ with $\pi(p) = g$. □

5.3 Proposition. *The quotient map π of G_f induces a one-to-one map from $G_f \backslash \mathcal{R}(\mathcal{T})$ to $\mathcal{G}(A, \mathrm{i}_A)$.*

We prove Proposition 5.3 in three steps and prepend a preliminary observation, which will be used later again:

5.4 Lemma. *For any two reduced p and q in $\mathcal{T}$ of finite length, such that $\pi(p) = \pi(q)$, there is an isometry $g \in G_f$ with $q = g(p)$.*

Proof. For two path a_1, b_1 of length one, the equality $\pi(a_1) = \pi(b_1)$ is equivalent with the existence of an isometry $g \in G_f$ such that $b_1 = g(a_1)$, since π is the quotient map of the action of G_f on $\mathcal{T}$.

For paths of length greater one, π still induces a projection from the cover to the quotient graph. Different G orbits of such objects may however project to the same quotient object. We prove now that G_f is large enough (for the purpose of equation (5.4)).

Inductively we assume the existence of $g_B \in G_f$ such that

$$(b_1, \ldots, b_m) = g_B(a_1, \ldots, a_m).$$

Both $(b_1, \ldots, b_{m+1})$ and $(b_1, \ldots, b_m, g_B(a_{m+1}))$ are reduced paths, hence for $E_R := \{b_{m+1}, \overline{b_{m+1}}, g_B(a_{m+1}), \overline{g_B(a_{m+1})}\}$ one has $E_R \cap \{b_1, \ldots, b_m\} = \emptyset$ by injectivity of reduced paths in trees.

Therefore $b_1, \dots, b_m$ is a path in the connected component at $o(b_{m+1})$ in $\mathcal{T} \setminus E_R$ and Lemma 3.11 provides an isometry $g_N \in G_f$ with $g_N(b_i) = b_i$ for all $1 \le i \le m$, and with $g_N(g_B(a_{m+1})) = b_{m+1}$. This shows Hence $g_N g_B(a_i) = b_i$ for all $1 \le i \le m+1$. $\square$

It is not obvious if one can extend the property of Lemma 5.4 to paths of infinite length, because an infinite product of group elements can not be defined without some notion of convergence. The solution which will be given here, makes use of a topological argument. First the statement of Lemma 5.4 shall be modified slightly.

Step 1 (Proof of Proposition 5.3). Suppose there are two reduced paths $p, q \in \mathcal{R}(\mathcal{T})$ such that $\pi(p) = \pi(q)$ and $p[i, i+1] = q[i, i+1]$ for all $i \le 0$. Then for every natural number $K \ge 0$ there exists an isometry $h \in G_f$ satisfying $h(p[i, i+1]) = q[i, i+1]$ for all $i \le K$.

Proof. We can use exactly the same arguments as in Lemma 5.4. For the root of induction ($K = 0$) we take $h := \mathrm{Id}|_{\mathcal{T}}$. For a step we assume the existence of an isometry $g_B \in G_f$ such that

$$g_B(p[i, i+1]) = q[i, i+1] \quad \text{for all} \quad i \le K-1.$$

The infinite path $L := (\dots, q[K-2, K-1], q[K-1, K])$ has no edges of $E_R := \left\{ g_B(p[K, K+1]), \overline{g_B(p[K, K+1])}, q[K, K+1], \overline{q[K, K+1]} \right\}$, since $g_B(p)$ and q are reduced paths (by injectivity of reduced paths in trees).

Thus L is a path in the connected component of $g_B(p(K)) = q(K)$ in the graph $\mathcal{T} \setminus E_R$. Then by Lemma 3.11 there is an isometry $g_N \in G_f$ with $g_N(L) = L$ and $g_N(g_B(p[K, K+1])) = q[K, K+1]$. The group element $g_N \circ g_B \in G_f$ has the desired property to complete an induction step for K. $\square$

Step 2. For any two reduced paths $p, q \in \mathcal{R}(\mathcal{T})$ with $\pi(p) = \pi(q)$ and $p(i) = q(i)$ for all $i \le 1$, there is an isometry $h \in G_f$ such that $q = h(p)$.

Proof. We can choose by the previous step a sequence of isometries $\{g_k\}_{k \in \mathbb{N}}$ of G_f with

$$g_k(p(j)) = q(j) \quad \text{for all} \quad j \le k.$$

For each $k \in \mathbb{N}$ we define a subset of G_f by

$$G_k := \{l \in G_f : l(p(j)) = g_k(p(j)) \quad \text{for all} \quad j \leq k\}.$$

Observe that the each G_k is non-empty, since $g_k \in G_k$ for all $k \in \mathbb{N}$. Note also, that $G_0 \supset G_1 \supset G_2 \supset \cdots$, because

$$\begin{aligned} G_{k+1} &= \{l \in G_f : l(p(j)) = g_{k+1}(p(j)) \quad \text{for all} \quad j \leq k+1\} \\ &\subset \{l \in G_f : l(p(j)) = g_{k+1}(p(j)) \quad \text{for all} \quad j \leq k\} = G_k, \end{aligned}$$

as $g_{k+1}p(j) = q(j) = g_k p(j)$ for all $j \leq k$.

Each of these sets can be written as $G_k = g_k \left(\bigcap_{j \leq k} (G_f)_{p(j)}\right)$, an intersection of vertex stabilizers translated by g_k. Vertex stabilizers are compact (cf. Corollary 3.16), so the intersection is compact, thus the translate G_k, too. In particular G_1 is compact and all $G_k's$ are closed, since G_f is Hausdorff. $\{G_k\}_{k \in \mathbb{N}}$ is therefore a family of closed sets in a compact space, where every finite sub family has non-void intersection. With this property, Theorem 5,1 in [18] states

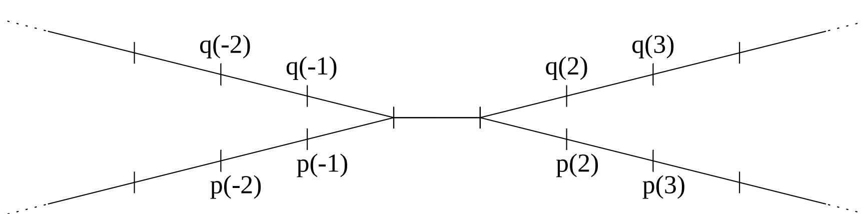
$$\emptyset \neq G_1 \cap G_2 \cap G_3 \cap \cdots.$$

We choose h out of this infinite intersection. For all $n \leq 1$ holds $h(p(n)) = p(n) = q(n)$ since $h \in G_1$. For $n \geq 2$ is $h(p(n)) = g_n(p(n)) = q(n)$ since $h \in G_n$. Altogether $h(p) = q$. □

Last Step. Given two reduced paths $p, q \in \mathcal{R}(\mathcal{T})$ with $\pi(p) = \pi(q)$, there is an isometry $h \in G_f$ such that $q = h(p)$.

Proof. Since $\pi(p[0,1]) = \pi(q[0,1])$ we may assume that $p[0,1] = q[0,1]$ (cf. Figure 5.2). By the previous step there is an isometry $g_p \in G_f$ such that

Figure 5.2: Two geodesics p, q crossing at $p[0,1] = q[0,1]$



$$g_p(p(i)) = \begin{cases} p(i) & \text{for all} \quad i \leq 0 \\ q(i) & \text{for all} \quad i \geq 1. \end{cases}$$

Also, by a symmetry argument, there is an isometry $g_q \in G_f$ with

$$g_q\left(q(i)\right) = \begin{cases} p(i) & \text{for all} \quad i \leq 0 \\ q(i) & \text{for all} \quad i \geq 1. \end{cases}$$

Thus an evaluation gives $g_p(p(i)) = g_q(q(i))$ for all i. This amounts to $g_p(p) = g_q(q)$, hence $q = g_q^{-1} g_p(p)$. □

5.5 Corollary. *The map* $\pi : G_f \backslash \mathcal{R}(\mathcal{T}) \longrightarrow \mathcal{G}(A, \mathrm{i}_A)$ *induced by the quotient map of the action of* G_f *on* $\mathcal{T}$ *is a* $\mathbb{Z}$*-equivariant one-to-one correspondence.*

Proof. Besides the statements of lemmas 5.1, and 5.2 and Proposition 5.3 it remains to show that the diagram

$$\begin{array}{ccc} G_f\backslash\mathcal{R}(\mathcal{T}) & \xrightarrow{L_\mathcal{R}} & G_f\backslash\mathcal{R}(\mathcal{T}) \\ \pi\big\downarrow & & \big\downarrow\pi \\ \mathcal{G}(A, \mathrm{i}_A) & \xrightarrow[L]{} & \mathcal{G}(A, \mathrm{i}_A) \end{array}$$

commutes, i.e. $\pi \circ L_\mathcal{R} = L \circ \pi$. For all $i \in \mathbb{Z}$ and all reduced paths $r \in \mathcal{RT}$ is $\pi(L_\mathcal{R} r)[i, i+1] = \pi((L_\mathcal{R} r)[i, i+1]) = \pi(r[i+1, i+2]) = (\pi r)[i+1, i+2]$ $= L(\pi r)[i, i+1]$. □

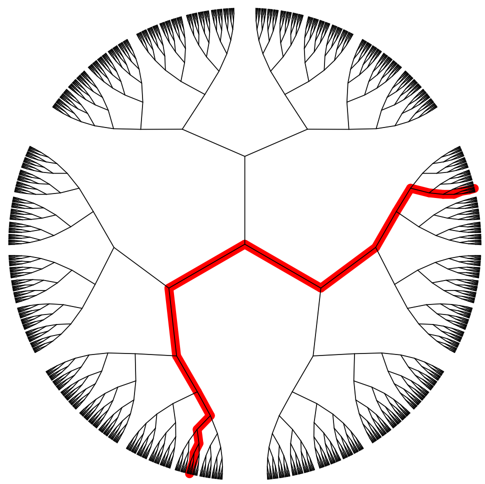
5.3 Description by border points and integers

The main point of this section is to prove a one-to-one correspondence. The claim is

$$\mathcal{RT} = (\mathcal{T}(\infty) \times \mathcal{T}(\infty)) \setminus \mathrm{diag} \times \mathbb{Z} \tag{5.5}$$

for $\mathrm{diag} = \{(\eta, \eta) \in \mathcal{T}(\infty) \times \mathcal{T}(\infty) : \eta \in \mathcal{T}(\infty)\}$. We also introduce a $\mathbb{Z}$-action on the right-hand side of above equation and show that the identification commutes with the $\mathbb{Z}$-actions on both sides (a $\mathbb{Z}$-action on $\mathcal{RT}$ has been introduced in the last section). A formula will be given for the action of isometries $h \in \mathrm{Is}\,(\mathcal{T})$ on triples of coordinates (η, ξ, n) that commutes with with the identification. Only in this section we prefer to call an element of $\mathcal{RT}$ a *geodesic*, rather than a bi-infinite reduced path. There is an illustration of such coordinates for geodesics in Figure 5.3. To give a prove of equation (5.5), some new notation will be

Figure 5.3: Visualization: A geodesic in a tree is determined (up to time parameterization) by a pair of distinct border points.



developed firstly. We introduce the maps

$$\alpha \quad : \quad \begin{array}{rcl} \mathcal{RT} & \longrightarrow & \mathcal{T}(\infty) \\ g & \longmapsto & [g(0), g(-1), g(-2), \ldots] \in \mathcal{R}_\infty / \sim \end{array}$$

$$\omega \quad : \quad \begin{array}{rcl} \mathcal{RT} & \longrightarrow & \mathcal{T}(\infty) \\ g & \longmapsto & [g(0), g(1), g(2), \ldots] \in \mathcal{R}_\infty / \sim . \end{array}$$

$\alpha(g)$ is called the *past* of g, $\omega(g)$ is called the *future* of g and g will be called a geodesic *from* the past *to* the future.

5.6 Definition (Projection to a geodesic). For each geodesic $g \in \mathcal{R}(\mathcal{T})$ with vertices $\mathrm{V}g := \{\ldots, g(-1), g(0), g(1), \ldots\}$ and each vertex $x \in \mathcal{T}$, the *projection of x to g* is defined as $\mathbf{p}(x, g) := g(k)$ such that $\mathrm{d}(x, g(k)) \leq \mathrm{d}(x, y)$ for all $y \in \mathrm{V}g$.

We will have to show, that there exists a unique vertex $\mathbf{p}(x, g)$, which complies with this demand. Existence is clear by well-ordering of $\mathbb{N}_0$. For uniqueness we may choose two candidates p, q for $\mathbf{p}(x, g)$.

The path $[x, p][p, z]$ is reduced for all $z \in \mathrm{V}g$, otherwise p was not closest to x in $\mathrm{V}g$. Similarly $[x, q][q, z]$ is reduced for all $z \in \mathrm{V}g$. Particularly, the paths

$[x,p][p,q]$ and $[x,q][q,p]$ are reduced. Lemma 1.25 can be used then to calculate

$$\begin{aligned} \mathrm{d}(x,p)+\mathrm{d}(p,q) &= \mathrm{d}(x,q) \\ \mathrm{d}(x,q)+\mathrm{d}(q,p) &= \mathrm{d}(x,p). \end{aligned}$$

The sum of these equations reads $\mathrm{d}(p,q)=0$, thus $p=q$.

For fixed vertices x,y, a function $G(x,y):\mathcal{RT}\mapsto\mathbb{Q}$ is defined as

$$G_g(x,y) := \frac{1}{2}\left(B_{\omega(g)}(x,y) - B_{\alpha(g)}(x,y)\right) \tag{5.6}$$

for all geodesics $g\in\mathcal{RT}$. B denotes the horocycle distance of Section 4.3. For any fixed vertex $x\in \mathrm{V}\mathcal{T}$, the map

$$\kappa_x: \begin{array}{ccccccc} \mathcal{RT} & \longrightarrow & \mathcal{T}(\infty) & \times & \mathcal{T}(\infty) & \times & \mathbb{Q} \\ g & \longmapsto & (\ \alpha(g), & & \omega(g), & & G_g\left(x,g(0)\right)\ ) \end{array} \tag{5.7}$$

will turn out to be suitable to prove equation (5.5).

We fix a vertex x and a geodesic g and write α for $\alpha(g)$ as well as ω for $\omega(g)$. As a consequence of Lemma 4.15 we note

$$B_\omega(g(k),g(l)) = -B_\alpha(g(k),g(l)) \quad (= l-k) \tag{5.8}$$

for all k,l in $\mathbb{Z}$, since we can define a geodesic $\overline{g}$ by $\overline{g}(i):=g(-i)$ for all $i\in\mathbb{Z}$. Then $\overline{g}(0),\overline{g}(1),\overline{g}(2),\ldots\in\alpha$ and $B_\alpha(g(k),g(l)) = B_\alpha(\overline{g}(-k),\overline{g}(-l)) = -l+k$. From equations (5.8) and (4.8) one deduces

$$\begin{aligned} & G_g(x,g(n)) + B_\alpha(g(m),g(n)) \\ &= \tfrac{1}{2}B_\omega(x,g(n)) - \tfrac{1}{2}B_\alpha(x,g(n)) + B_\alpha(g(m),g(n)) \\ &= \tfrac{1}{2}\left(B_\omega(x,g(n)) + B_\omega(g(n),g(m)) - B_\alpha(x,g(n)) - B_\alpha(g(n),g(m))\right) \\ &= \tfrac{1}{2}\left(B_\omega(x,g(m)) - B_\alpha(x,g(m))\right) \quad = \quad G_g(x,g(m)) \end{aligned} \tag{5.9}$$

for all $n,m\in\mathbb{Z}$.

To do some explicit calculation of G, we need a connection between the projection $\mathbf{p}(\cdot,\cdot,\cdot)$ to a ray and the projection $\mathbf{p}(\cdot,\cdot)$ to a geodesic. We show in this paragraph

$$\begin{aligned} \mathbf{p}(x,\mathbf{p}(x,g),\alpha) &= \mathbf{p}(x,g) \quad \text{and} \\ \mathbf{p}(x,\mathbf{p}(x,g),\omega) &= \mathbf{p}(x,g) \end{aligned} \tag{5.10}$$

in two steps. First, there is the distance equality

$$\mathrm{d}(x,\mathbf{p}(x,\mathbf{p}(x,g),\alpha)) = \mathrm{d}(x,\mathbf{p}(x,g)).$$

From equation (4.4) one has $\mathrm{d}\left(x, \mathbf{p}\left(x, \mathbf{p}(x,g), \omega\right)\right) = \min_{y \in [\mathbf{p}(x,g),\omega)} \mathrm{d}(x,y)$. This gives

$$\min_{y \in [\mathbf{p}(x,g),\omega)} \mathrm{d}(x,y) \leq \mathrm{d}\left(x, \mathbf{p}(x,g)\right)$$

as well as

$$\min_{y \in [\mathbf{p}(x,g),\omega)} \mathrm{d}(x,y) \geq \min_{y \in \mathrm{V}g} \mathrm{d}(x,y) = \mathrm{d}\left(x, \mathbf{p}(x,g)\right).$$

Second, one can show, that

$$\mathbf{p}(x,g) \in [x, \alpha).$$

This assertion is equivalent with $[x, \mathbf{p}(x,g)][\mathbf{p}(x,g), \alpha)$ being reduced. If this ray was not reduced, there would be a vertex $y \neq \mathbf{p}(x,g)$ in the path $[x, \mathbf{p}(x,g)]$, that appears in the ray $[\mathbf{p}(x,g), \alpha)$. The inequality $\mathrm{d}(x,y) < \mathrm{d}(x, \mathbf{p}(x,g))$ contradicts then $\mathrm{d}(x, \mathbf{p}(x,g)) = \min_{z \in \mathrm{V}g} \mathrm{d}(x,z)$, because $[\mathbf{p}(x,g), \alpha) \subset \mathrm{V}g$. This proves equation (5.10) for α. The proof of the second equation for ω goes verbatim.

Using these results, one can calculate the contributing horocycle distances between x and $\mathbf{p}(x,g)$ as

$$B_\omega(x, \mathbf{p}(x,g)) = B_\alpha(x, \mathbf{p}(x,g)) = \mathrm{d}(x, \mathbf{p}(x,g)), \tag{5.11}$$

because

$$\begin{aligned} B_\omega(x, \mathbf{p}(x,g)) \quad &= \quad \mathrm{d}(x, \mathbf{p}(x, \mathbf{p}(x,g), \omega)) - \mathrm{d}(\mathbf{p}(x,g), \mathbf{p}(\mathbf{p}(x,g), x, \omega)) \\ \overset{(5.10)}{=} \quad \mathrm{d}(x, \mathbf{p}(x,g)) - \mathrm{d}(\mathbf{p}(x,g), \mathbf{p}(x,g)) \quad &= \quad \mathrm{d}(x, \mathbf{p}(x,g)) \end{aligned}$$

by equation (5.10). Similarly one argues for α. Equation 5.11 permits us finally to evaluate the function G at a first vertex, at the projection $\mathbf{p}(x,g)$.

$$G_g(x, \mathbf{p}(x,g)) = 0. \tag{5.12}$$

Other vertices will be evaluated using this same formula and equations (5.9) and (5.8): Assuming $\mathbf{p}(x,g) = g(n)$ one has for vertices $g(m)$ of g $(m \in \mathbb{Z})$

$$\begin{aligned} G_g(x, g(m)) \quad &= \quad G_g(x, g(n)) + B_\alpha(g(m), g(n)) \\ = \quad G_g(x, \mathbf{p}(x,g)) + m - n \quad &= \quad m - n. \end{aligned} \tag{5.13}$$

Since g is an injective path, $[g(0), \alpha)$ and $[g(0), \omega)$ obviously have no infinite intersection hence $\alpha(g) \neq \omega(g)$. We can write therefore:

5.7 Proposition. *The map κ_x is for each $x \in \mathrm{V}\mathcal{T}$ a map from $\mathcal{R}(\mathcal{T})$ to $(\mathcal{T}(\infty) \times \mathcal{T}(\infty)) \setminus \mathrm{diag} \times \mathbb{Z}$.*

It is not too difficult to see, that the map $g \mapsto (\alpha(g), \omega(g))$ is onto $(\mathcal{T}(\infty) \times \mathcal{T}(\infty)) \setminus \operatorname{diag}$, if g varies in $\mathcal{R}(\mathcal{T})$. For each pair $\eta, \xi \in \mathcal{T}(\infty)$ such that $\eta \neq \xi$ we can form the intersection $[x, x_n] = [x, \eta) \cap [x, \xi)$ (cf. Definition 4.4). The $\mathbb{Z}$-sequence

$$g(i) := \begin{cases} x_\eta(n-i) & \text{for} \quad i \leq 0 \\ x_\xi(n+i) & \text{for} \quad i > 0 \end{cases}$$

defines a geodesic $g \in \mathcal{R}(\mathcal{T})$, because $[x, x_n]$ contains all vertices, which are common to x_η and to x_ξ. Obviously $\alpha(g) = \eta$ and $\omega(g) = \xi$. In the previous section we introduced an action of $\mathbb{Z}$ on $\mathcal{R}(\mathcal{T})$ by

$$L_{\mathcal{R}}(n)g := h \tag{5.14}$$

such that $h(i) = g(i+n)$ for all $i \in \mathbb{Z} = \mathrm{V}h$ and for all $n \in \mathbb{Z}$. By definition of past and future we obtain $\alpha(h) = \alpha(L_{\mathcal{R}}(n)g) = \alpha(g)$ and also $\omega(h) = \omega(L_{\mathcal{R}}(n)g) = \omega(g)$. Hence by (5.6)

$$G_h = G_{L_{\mathcal{R}}(n)g} = G_g. \tag{5.15}$$

For any pair $\eta \neq \xi$ we choose a geodesic g from η to ξ, say $\mathbf{p}(x, g) = g(k)$. Then

$$G_h(x, h(0)) = G_g(x, (L_{\mathcal{R}}(n)g)(0)) = G_g(x, g(n)) = n - k \tag{5.16}$$

by equation (5.13) and obtain

$$\kappa_x(h) = \kappa_x\left(L_{\mathcal{R}}(n)g\right) = (\eta, \xi, n-k) \tag{5.17}$$

for all $n \in \mathbb{Z}$. This shows surjectivity of κ_x.

5.8 Proposition. *The map* $\kappa_x : \mathcal{R}(\mathcal{T}) \longrightarrow (\mathcal{T}(\infty) \times \mathcal{T}(\infty)) \setminus \operatorname{diag} \times \mathbb{Z}$ *is a surjective map for all* $x \in \mathrm{V}\mathcal{T}$.

The section shall continued with a proof of injectivity of κ_x. Assume $g, h \in \mathcal{R}(\mathcal{T})$, $\alpha(g) = \alpha(h)$ and $\omega(g) = \omega(h)$. $[g(0), \omega)$ and $[h(0), \omega)$ have infinite intersection, hence $g(k) = h(l)$ for some $k, l \in \mathbb{Z}$. Then $[g(k), \omega) = [h(l), \omega)$ shows $g(k+n) = h(l+n)$ for all $n \geq 0$, whereas $[g(k), \alpha) = [h(l), \alpha)$ implies that $g(k+n) = h(l+n)$ for all $n \leq 0$. This shows $h(n) = g(n+m)$ for $m := k - l$ and all $n \in \mathbb{Z}$, i.e. $h = L_{\mathcal{R}}(m)g$.

If in addition we suppose equality in the third coordinate of the images of g and h under κ_x and assume $\mathbf{p}(x, g) = j$, then

$$\begin{aligned} 0 &= G_h(x, h(0)) - G_g(x, g(0)) \\ &= G_{L_{\mathcal{R}}(m)g}(x, (L_{\mathcal{R}}(m)g)(0)) - G_g(x, g(0)) \\ &= G_g(x, g(m)) - G_g(x, g(0)) \quad = \quad (m-j) - (0-j) \\ &= m \end{aligned}$$

by equation (5.13). This shows $h = L_{\mathcal{R}}(0)g = g$.

5.9 Proposition. *The map $\kappa_x : \mathcal{R}(\mathcal{T}) \longrightarrow (\mathcal{T}(\infty) \times \mathcal{T}(\infty)) \setminus \mathrm{diag} \times \mathbb{Z}$ is a bijective map for all $x \in \mathrm{V}\mathcal{T}$.*

Finally, the group actions on the geodesic space $\mathcal{GT}$ are transferred to the new coordinates in such a way that they commute with identifications κ_x. A $\mathbb{Z}$-action can be defined on $(\mathcal{T}(\infty) \times \mathcal{T}(\infty)) \setminus \mathrm{diag} \times \mathbb{Z}$ by

$$L_B(\eta, \xi, n) := (\eta, \xi, n+1). \tag{5.18}$$

for all $(\eta, \xi, n) \in (\mathcal{T}(\infty) \times \mathcal{T}(\infty)) \setminus \mathrm{diag} \times \mathbb{Z}$.

5.10 Corollary. *For each $x \in \mathrm{V}\mathcal{T}$, the map κ_x is a $\mathbb{Z}$-equivariant identification.*

Proof. It remains to prove that the diagram

$$\begin{array}{ccc} \mathcal{RT} & \xrightarrow{\kappa_x} & (\mathcal{T}(\infty) \times \mathcal{T}(\infty)) \setminus \mathrm{diag} \times \mathbb{Z} \\ {\scriptstyle L_{\mathcal{R}}}\downarrow & & \downarrow{\scriptstyle L_B} \\ \mathcal{RT} & \xrightarrow[\kappa_x]{} & (\mathcal{T}(\infty) \times \mathcal{T}(\infty)) \setminus \mathrm{diag} \times \mathbb{Z} \end{array}$$

commutes, i.e. $L_B \circ \kappa_x = \kappa_x \circ L_{\mathcal{R}}$. Since as above $\alpha(L_{\mathcal{R}}g) = \alpha(g)$, $\omega(L_{\mathcal{R}}g) = \omega(g)$ and $G_{L_{\mathcal{R}}g} = G_g$, we are done if we show $G_g(x, g(1)) = G_g(x, g(0)) + 1$. But this follows from equation (5.13). □

5.11 Proposition. *The action of an isometry $h \in \mathrm{Is}\,(\mathcal{T})$ on a triple $(\eta, \xi, n) \in (\mathcal{T}(\infty) \times \mathcal{T}(\infty)) \setminus \mathrm{diag} \times \mathbb{Z}$ defined as*

$$h(\eta, \xi, n) := \big(h\eta, h\xi, n + G_{h(g)}(x, hx)\big) \tag{5.19}$$

commutes with the identification $\kappa_x : \mathcal{R}(\mathcal{T}) \longrightarrow (\mathcal{T}(\infty) \times \mathcal{T}(\infty)) \setminus \mathrm{diag} \times \mathbb{Z}$ for any $x \in \mathrm{V}\mathcal{T}$.

Proof. A geodesic g shall be chosen. As above we write α for $\alpha(g)$ and ω for $\omega(g)$. Simply by definition one obtains $\kappa_x \circ h(g) = (h\alpha, h\omega, G_{h(g)}(x, hg(0)))$. The fact $\alpha(hg) = h(\alpha(g)) = h(\alpha)$ is easily checked. Similarly $\omega(hg) = h(\omega)$. On the other hand, to verify above formula, one has to compare $\kappa_x \circ h(g)$ with $h \circ \kappa_x(g) = h(\alpha, \omega, G_g(x, g(0))) = (h\alpha, h\omega, G_g(x, g(0)) + G_{h(g)}(x, hx))$. The first two coordinates are correct already, so it remains to prove $G_g(x, g(0)) + G_{h(g)}(x, hx) = G_{h(g)}(x, hg(0))$. This can be written as

$$\begin{aligned}
& G_g(x, g(0)) + G_{h(g)}(x, hx) \\
= \quad & \tfrac{1}{2}\left(B_\omega(x, g(0)) - B_\alpha(x, g(0))\right) + \tfrac{1}{2}\left(B_{h(\omega)}(x, hx) - B_{h(\alpha)}(x, hx)\right) \\
\overset{(4.9)}{=} \quad & \tfrac{1}{2}\left(B_{h(\omega)}(x, hx) + B_{h(\omega)}(hx, hg(0)) - B_{h(\alpha)}(x, hx) - B_{h(\alpha)}(hx, hg(0))\right) \\
\overset{(4.8)}{=} \quad & \tfrac{1}{2}\left(B_{h(\omega)}(x, hg(0)) - B_{h(\alpha)}(x, hg(0))\right) \\
= \quad & G_{h(g)}\left(x, hg(0)\right).
\end{aligned}$$

□

5.4 α-dimensional densities on the border

In Section 4.5 the vector space $S(\mathcal{T}(\infty))$ of locally constant functions on the border of a tree has been introduced. The *dual space* $S(\mathcal{T}(\infty))^*$ of this vector space consists of all linear functions, called *functionals*, $\Lambda : S(\mathcal{T}(\infty)) \to \mathbb{C}$. This space is a vector space itself as the range of functionals is $\mathbb{C}$, a vector space. An α-dimensional density μ is a function

$$\mu : \begin{array}{ccc} X & \longrightarrow & S(\mathcal{T}(\infty))^* \\ x & \longmapsto & \mu_x \end{array}$$

such that

$$\mu_x(\varphi) = \mu_y(\varphi e^{-\alpha B(x,y)})$$

holds for all $x, y \in \mathrm{V}\mathcal{T}, \varphi \in S(\mathcal{T}(\infty))$. Note that $\omega \mapsto B_\omega(x, y)$ is locally constant (Lemma 4.21). The vector space of all α-dimensional densities is denoted by $\mathcal{D}_\alpha$. Two linear maps will be used for further analysis:

a) $Q : \mathcal{D}_\alpha \longrightarrow \mathbb{C}\mathrm{E}A$
   defined as $(Q\mu)(a) := \mu_{\mathrm{o}(a)}(\chi_{\Omega(a)})$ for all edges $a \in \mathrm{E}\mathcal{T}$. We set $\Omega(a) := \Omega(\mathrm{o}(a), \mathrm{t}(a))$.

b) $R : \mathbb{C}(\mathrm{E}A) \longrightarrow \mathbb{C}(\mathrm{E}A)$

defined as $RF(a) := \sum_{\substack{\mathrm{o}(b)=\mathrm{t}(a)\\ b\neq\overline{a}}} F(b)$ for all edges $a \in \mathrm{E}\mathcal{T}$.

We introduce the linear subspace E_α of $\mathbb{C}(\mathrm{E}A)$ formed by all functions F, which are solutions of the linear equation $(R - e^\alpha \cdot \mathrm{Id}_{\mathbb{C}(Y)})F = 0$:

$$E_\alpha := \{F \in \mathbb{C}(\mathrm{E}A) : RF = e^\alpha F\}.$$

When two edges $a, b \in \mathrm{E}\mathcal{T}$ satisfy $\mathrm{o}(b) = \mathrm{t}(a)$ and $b \neq \overline{a}$ and $\xi \in \Omega(b)$, then $[\mathrm{o}(a), \mathrm{t}(a)][\mathrm{t}(a), \xi)$ is reduced. Hence $\mathrm{o}(a) = \mathrm{o}(a)_\xi(0)$, $\mathrm{t}(a) = \mathrm{o}(a)_\xi(1)$ and we get by Corollary 4.16 $B_\xi(\mathrm{o}(b), \mathrm{o}(a)) = B_\xi(\mathrm{o}(a)_\xi(1), \mathrm{o}(a)_\xi(0)) = 0 - 1 = -1$. So prepared we find for all $\mu \in \mathcal{D}_\alpha$ and all $a \in \mathrm{E}\mathcal{T}$

$$\begin{aligned} R(Q\mu)(a) &= \sum_{\substack{\mathrm{o}(b)=\mathrm{t}(a)\\ b\neq\overline{a}}} (Q\mu)(b) = \sum_{\substack{\mathrm{o}(b)=\mathrm{t}(a)\\ b\neq\overline{a}}} \mu_{\mathrm{o}(b)}(\chi_{\Omega(b)}) \\ &= \sum_{\substack{\mathrm{o}(b)=\mathrm{t}(a)\\ b\neq\overline{a}}} \mu_{\mathrm{o}(a)}(\chi_{\Omega(b)} e^{-\alpha B(\mathrm{o}(b),\mathrm{o}(a))}) \\ &= e^\alpha \sum_{\substack{\mathrm{o}(b)=\mathrm{t}(a)\\ b\neq\overline{a}}} \mu_{\mathrm{o}(a)}(\chi_{\Omega(b)}) = e^\alpha \mu_{\mathrm{o}(a)}\Big(\sum_{\substack{\mathrm{o}(b)=\mathrm{t}(a)\\ b\neq\overline{a}}} \chi_{\Omega(b)}\Big) \\ &\overset{(4.13)}{=} e^\alpha \mu_{\mathrm{o}(a)} \chi_{\Omega(a)} = e^\alpha (Q\mu)(a). \end{aligned}$$

In other words, the image of the space of α-dimensional densities $\mathcal{D}_\alpha$ under Q is a subspace of E_α:

$$Q(\mathcal{D}_\alpha) \subset E_\alpha. \tag{5.20}$$

5.12 Proposition. *The map $Q : \mathcal{D}_\alpha \longrightarrow E_\alpha$ is an isomorphism of vector spaces.*

Step 1 (Proof of Proposition 5.12). Q is injective.

Proof. Let $\mu \in \mathcal{D}_\alpha$ and suppose $(Q\mu)(a) = 0$ for all $a \in \mathrm{E}\mathcal{T}$. For all those edges a, where $[x, \mathrm{o}(a)][\mathrm{o}(a), \mathrm{t}(a)]$ is reduced ($n := \mathrm{d}(x, \mathrm{o}(a))$), one has

$$\begin{aligned} \mu_x(a) &= \mu_{\mathrm{o}(a)}(\chi_{\Omega(a)} e^{-\alpha B(x,\mathrm{o}(a))}) = e^{-n\alpha} \mu_{\mathrm{o}(a)}(\chi_{\Omega(a)}) \\ &= e^{-n\alpha} Q\mu(a). \end{aligned}$$

For all $\xi \in \Omega(a) = \Omega(x, \mathrm{t}(a)) \subset \Omega(x, \mathrm{o}(a))$ is $x = x_\xi(0)$ as well as $\mathrm{o}(a) = x_\xi(n)$, hence by Corollary 4.16 $B_\xi(x, \mathrm{o}(a)) = B_\xi(x_\xi(0), x_\xi(n)) = n$. Now since $\{\chi_{\Omega(a)}\}_{\substack{[x,\mathrm{o}(a)][\mathrm{o}(a),\mathrm{t}(a)]\\ \text{is reduced}}} = \{\chi_{\Omega(x,z)}\}_{\mathrm{d}(x,z)\geq 1}$ generates $S(\mathcal{T}(\infty))$ we get $\mu_x = 0$. This holds for all $x \in \mathrm{V}\mathcal{T}$, whence $\mu = 0$. □

Step 2. For each $F \in \mathrm{E}_\alpha$ and each $x \in \mathrm{V}\mathcal{T}$ there is a functional $\lambda_x \in S(\mathcal{T}(\infty))^+$, such that

$$\lambda_x(\chi_{\Omega(a)}) = e^{-(\mathrm{d}(x,\mathrm{o}(a))\alpha} F(a) \tag{5.21}$$

for all $a \in E_n := \{a_n \in \mathrm{E}\mathcal{T} : a_1, \ldots, a_n \text{ reduced, } \mathrm{o}(a_1) = x\}$ $(n \geq 1)$.

Proof. For each function space $\mathcal{F}_0 := \{c_x \chi_{\mathcal{T}(\infty)} : c_x \in \mathbb{C}\}$ respectively $\mathcal{F}_n := \{\sum_{a \in E_n} c_a \chi_{\Omega(a)} : c_a \in \mathbb{C} \text{ for all } a \in \mathrm{E}_n\}$ $(n \geq 1)$, the sets of functions $\{\chi_{\mathcal{T}(\infty)}\}$ respectively $\{\chi_{\Omega(a)}\}_{a \in E_n}$ for $n \geq 1$ form a base. Thus we can define a functional λ_x^0 on $\mathcal{F}_0$ by $\lambda_x^0(\chi_{\mathcal{T}(\infty)}) := \sum_{a \in E_1} F(a)$. For $n \geq 1$ we define a functional λ_x^n on $\mathcal{F}_n$ by $\lambda_x^n(\chi_{\Omega(a)}) := e^{-(\mathrm{d}(x,\mathrm{o}(a))\alpha} F(a)$ for all $a \in E_n$.

By (4.19), every function $\varphi \in S(\mathcal{T}(\infty))$ is in one of the function spaces $\mathcal{F}_i$. If n is minimal such that $\varphi \in \mathcal{F}_n$, we say that φ is of *minimal type* n and put

$$\lambda_x(\varphi) := \lambda_x^n(\varphi).$$

If φ is of minimal type n and $k \geq n$ we say that φ is of *type* k if $\varphi \in \mathcal{F}_k$ and

$$\lambda_x(\varphi) = \lambda_x^k(\varphi).$$

A function of minimal type n is of type n by definition. This is the root of induction for the following statement: A function of minimal type n is of type k for all $k \geq n$. Note that

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \cdots$$

by (4.13). Hence a function of type $k \geq 0$ is in $\mathcal{F}_k \subset \mathcal{F}_{k+1}$. Now if φ is of type 0 we can write $\varphi = c_x \chi_{\mathcal{T}(\infty)} = \sum_{a \in E_1} c_x \chi_{\Omega(a)}$.

$$\begin{aligned} \lambda_x^1(\varphi) &= \lambda_x^1(c_x \sum_{a \in E_1} \chi_{\Omega(a)}) = \sum_{a \in E_a} c_x \lambda_x^1(\chi_{\Omega(a)}) = c_x \sum_{a \in E_1} F(a) \\ &= c_x \lambda_x^0(\chi_{\mathcal{T}(\infty)}) = \lambda_x^0(c_x \chi_{\mathcal{T}(\infty)}) = \lambda_x^0(\varphi). \end{aligned}$$

Since φ is trivially of minimal type 0, this equals $\lambda_x(\varphi)$ by definition. We proved, that φ is of type 1, too. If φ is of type $k \geq 1$ then we can write

$\varphi = \sum_{a \in E_k} c_a \chi_{\Omega(a)} = \sum_{\substack{a_1,\ldots,a_{k+1} \\ \text{reduced}}} c_{a_k}\ \chi_{\Omega(a_{k+1})}$ and get (we take $\mathrm{o}(a_1) = x$)

$$\begin{aligned}
\lambda_x^{k+1}(\varphi) &= \lambda_x^{k+1}\Big(\sum_{\substack{a_1,\ldots,a_{k+1} \\ \text{reduced}}} c_{a_k}\chi_{\Omega(a_{k+1})}\Big) = \sum_{\substack{a_1\ldots,a_{k+1} \\ \text{reduced}}} c_{a_k}\lambda_x^{k+1}(\chi_{\Omega(a_{k+1})}) \\
&= \sum_{\substack{a_1\ldots,a_{k+1} \\ \text{reduced}}} c_{a_k} e^{-k\alpha} F(a_{k+1}) = \sum_{\substack{a_1\ldots,a_k \\ \text{reduced}}} c_{a_k} e^{-k\alpha} \sum_{\substack{a_k,a_{k+1} \\ \text{reduced}}} F(a_{k+1}) \\
&= \sum_{\substack{a_1\ldots,a_k \\ \text{reduced}}} c_{a_k} e^{-k\alpha} RF(a_k) = \sum_{a \in E_k} c_a e^{-k\alpha} RF(a) \\
&= \sum_{a \in E_k} c_a e^{-(k-1)\alpha} F(a) = \sum_{a \in E_k} c_a \lambda_x^k(\chi_{\Omega(a)}) = \lambda_x^k\Big(\sum_{a \in E_k} c_a \chi_{\Omega(a)}\Big) \\
&= \lambda_x^k(\varphi).
\end{aligned}$$

By assumption of induction this equals $\lambda_x(\varphi)$, hence φ is of type $k+1$.

For a prove of linearity of λ_x we may consider two functions $\varphi, \psi \in S(\mathcal{T}(\infty))$ and the sum $c\varphi + d\psi \in S(\mathcal{T}(\infty))$. We take M as the maximum of the minimal types of these three functions (they are then of type M) and get $\lambda_x(c\varphi + c\psi) = \lambda_x^M(c\varphi + c\psi) = c\lambda_x^M(\varphi) + d\lambda_x^M(\psi) = c\lambda_x(\varphi) + c\lambda_x(\psi)$. ☐

Step 3. For each $F \in E_\alpha$, the selection of functionals λ_x, $x \in \mathrm{V}\mathcal{T}$, defined in Step 2, forms an α-dimensional density λ.

Proof. $S(\mathcal{T}(\infty))$ is generated by $\{\chi_{\Omega(x,z)}\}_{\mathrm{d}(x,z) \geq N}$ for all $N \in \mathbb{N}_0$ as well as by $\{\chi_{\Omega(y,z)}\}_{\mathrm{d}(y,z) \geq N}$ for all $N \in \mathbb{N}_0$. We choose $N := \mathrm{d}(x,y) + 1$ in both cases. The spheres appearing in theses selections are all of the form $\Omega(a)$ such that both $[x, \mathrm{o}(a)][\mathrm{o}(a), \mathrm{t}(a)]$ and $[y, \mathrm{o}(a)][\mathrm{o}(a), \mathrm{t}(a)]$ are reduced.

Now for $\mathrm{d}(x, \mathrm{o}(a)) = n$ we get for all $\xi \in \Omega(a) = \Omega(x, \mathrm{t}(a)) = \Omega(y, \mathrm{t}(a))$ by Corollary 4.16 $B_\xi(x, \mathrm{o}(a)) = B_\xi(x_\xi(0), x_\xi(n)) = n$ and similarly $B_\xi(y, \mathrm{o}(a)) = \mathrm{d}(y, \mathrm{o}(a))$. Thus $B_\xi(x,y) = B_\xi(x, \mathrm{o}(a)) + B_\xi(\mathrm{o}(a), y) = \mathrm{d}(x, \mathrm{o}(a)) - \mathrm{d}(y, \mathrm{o}(a))$ and therefore

$$\begin{aligned}
\lambda_x(\chi_{\Omega(a)}) &= e^{-\mathrm{d}(x,\mathrm{o}(a))\alpha} F(a) = e^{-\mathrm{d}(y,\mathrm{o}(a))\alpha} F(a) e^{-\alpha B(x,y)} \\
&= \lambda_y(\chi_{\Omega(a)} e^{-\alpha B(x,y)}).
\end{aligned}$$

☐

Last Step (Proof of Proposition 5.12). $Q : \mathcal{D}_\alpha \longrightarrow E_\alpha$ has a right inverse, i.e. Q is surjective.

Proof. We take Q' the map, which assigns by Step 3 the α-dimensional λ to the function $F \in \mathrm{E}_\alpha$. Then

$$Q(Q'F)(a) \quad = \quad (Q'F)_{\mathrm{o}(a)}(\chi_{\Omega(a)}) \quad = \quad e^{-\mathrm{d}(\mathrm{o}(a),\mathrm{o}(a))\alpha}F(a) \quad = \quad F(a).$$

□

Observe, that $\mathrm{Is}\,(\mathcal{T})$ acts on the dual space $S\,(\mathcal{T}(\infty))^*$. In Section 4.5 we saw that $\mathrm{Is}\,(\mathcal{T})$ acts on $S\,(\mathcal{T}(\infty))$. Now for an isometry h we define a functional $h * \Lambda$ by

$$h * \Lambda(\varphi) := \Lambda(h^{-1} * \varphi)$$

for all $\varphi \in S\,(\mathcal{T}(\infty))$. Linearity translates from linearity of $\Lambda \in S\,(\mathcal{T}(\infty))^*$. Given a subgroup $H < \mathrm{Is}\,(\mathcal{T})$, we call μ an α-dimensional density *for* H, if the diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu} & S\,(\mathcal{T}(\infty))^* \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[\mu]{} & S\,(\mathcal{T}(\infty))^* \end{array}$$

commutes for all $h \in H$, i.e. $\mu_{hx} = h * \mu_x$. $\mathcal{D}^H_\alpha$ is the set of α-dimensional densities for H.

Since $\mu_{hx} = h * \mu_x$ is equivalent with $h^{-1} * \mu_{hx} = \mu_x$, the space $\mathcal{D}^H_\alpha$ can be seen as the fixed vectors under an action of the group H on $\mathcal{D}$ given by

$$(h_*\mu)_x := h * (\mu_{h^{-1}x})$$

for isometries $h \in \mathrm{Is}\,(\mathcal{T})$ and α-dimensional densities μ. This assignment has to be verified as an action. It has to be proved first, that $h_*\mu$ is an α-dimensional density. For any $\varphi \in S\,(\mathcal{T}(\infty))$ one has

$$\begin{aligned} (h_*\mu)_x(\varphi) \quad &= \quad h * (\mu_{h^{-1}x})(\varphi) \quad = \quad \mu_{h^{-1}x}(h^{-1} * \varphi) \\ &= \quad \mu_{h^{-1}y}((h^{-1} * \varphi)e^{-\alpha B(h^{-1}x,h^{-1}y)}) \\ &\overset{(+)}{=} \quad \mu_{h^{-1}y}((h^{-1} * \varphi)(h^{-1} * e^{-\alpha B(x,y)})) \\ &= \quad h * (\mu_{h^{-1}y})(\varphi e^{-\alpha B(x,y)}) \quad = \quad (h_*\mu)_y(\varphi e^{-\alpha B(x,y)}). \end{aligned}$$

Equality at (+) follows from equation (4.9) as $B_\xi(h^{-1}x, h^{-1}y) = B_{h\xi}(x,y) = h^{-1} * B_\xi(x,y)$. The action of $\mathrm{Is}\,(\mathcal{T})$ is indeed a group action. For two isometries g, h we get

$$\begin{aligned} h_*(g_*\mu)_x \quad &= \quad h * ((g_*\mu)_{h^{-1}x}) \quad = \quad h * (g * \mu_{g^{-1}h^{-1}x}) \\ &= \quad hg * \mu_{(hg)^{-1}x} \quad = \quad ((hg)_*\mu)_x \end{aligned}$$

for all $x \in \mathrm{V}\mathcal{T}$. Similarly one argues for $\mathrm{Id} \in \mathrm{Is}(\mathcal{T})$.

Q is an $\mathrm{Is}(\mathcal{T})$-equivariant map. We show $h*(Q\mu) = Q(h_*\mu)$ for all isometries h and α-dimensional densities μ. Assume $a \in \mathrm{E}\mathcal{T}$ is an edge.

$$\begin{array}{ccccccc} Q(h_*\mu)(a) & = & (h_*\mu)_{\mathrm{o}(a)}(\chi_{\Omega(a)}) & = & \mu_{\mathrm{o}(h^{-1}a)}(h^{-1} * \chi_{\Omega(a)}) & & \\ \overset{(*)}{=} & \mu_{\mathrm{o}(h^{-1}a)}(\chi_{\Omega(h^{-1}a)}) & = & (Q\mu)(h^{-1}a) & = & h * (Q\mu)(a). \end{array} \tag{5.22}$$

The step (*) is correct, since $h^{-1} * \chi_{\Omega(a)}(\xi) = \chi_{\Omega(a)}(h\xi) = \xi_{h^{-1}\Omega(a)}(\xi) \overset{(4.16)}{=} \chi_{\Omega(h^{-1}a)}(\xi)$.

Observe once more that $\mathcal{D}_\alpha^H$, the set of α-dimensional densities for H is the set of H-fixed vectors in $\mathcal{D}_\alpha$. That is $h_*\mu = \mu$ for all $h \in H$ if and only if $\mu \in \mathcal{D}_\alpha^H$. We introduce the set of H-invariant functions $E_\alpha^H := \{F \in E_\alpha : h * F = F \text{ for all } h \in H\}$, the action given by $h * F(a) := F(h^{-1}a)$ for all edges $a \in \mathrm{E}\mathcal{T}$. The set E_α^H, consisting of functions constant on the orbits of H, is clearly a vector space.

5.13 Corollary. *The map $Q : \mathcal{D}_\alpha \longrightarrow E_\alpha$ induces an isomorphism between H-fixed α-dimensional densities and H-invariant functions of E_α, for every subgroup $H < \mathrm{Is}(\mathcal{T})$.*

Proof. By Proposition 5.12, Q is invertible. Thus one can write $Q^{-1}(E_\alpha^H) = \{Q^{-1}F \in \mathcal{D}_\alpha : h * F = F\} \overset{F=Q\mu}{=} \{\mu \in \mathcal{D}_\alpha : h * (Q\mu) = Q\mu\}$. As Q is an $\mathrm{Is}(\mathcal{T})$-equivariant map by equation (5.22), this equals

$$\{\mu \in \mathcal{D}_\alpha : Q(h_*\mu) = Q\mu\} \quad = \quad \{\mu \in \mathcal{D}_\alpha : h_*\mu = \mu\} \quad = \quad \mathcal{D}_\alpha^H.$$

□

We want to sort out functionals, that have an interpretation as integrals on $S(\mathcal{T}(\infty))$, so one needs a notion some of positivity. We set

$$S(\mathcal{T}(\infty))^+ := \{\varphi \in S(\mathcal{T}(\infty)) : \varphi(\xi) \geq 0 \text{ for all } \xi \in \mathcal{T}(\infty)\} \tag{5.23}$$

and say an α-dimensional density μ is *positive*, if $\mu_x(\varphi) \geq 0$ for all $x \in \mathrm{V}\mathcal{T}$ and all $\varphi \in S(\mathcal{T}(\infty))^+$.

$$\mathcal{D}_\alpha^+ := \{\mu \in \mathcal{D}_\alpha : \mu \text{ is positive }\}$$

is called the *real cone of positive α- dimensional densities.* We define also

$$E_\alpha^+ := \{F \in E_\alpha : F(e) \geq 0 \text{ for all } e \in \mathrm{E}\mathcal{T}\}$$

and call these functions *positive functions*.

5.14 Corollary. *The map $Q : \mathcal{D}_\alpha \longrightarrow E_\alpha$ induces a bijection between positive α-dimensional densities and positive functions in E_α.*

Proof. For all $\mu \in \mathcal{D}_\alpha^+$ and edges $a \in \mathrm{E}\mathcal{T}$ one has $Q(\mu)(a) = \mu_{\mathrm{o}(a)}(\chi_{\Omega(a)}) \geq 0$, since $\chi_{\Omega(a)} \in S\left(\mathcal{T}(\infty)\right)^+$. Therefore $Q(\mathcal{D}_\alpha^+) \subset E_\alpha^+$. Conversely, for $\varphi \in S\left(\mathcal{T}(\infty)\right)^+$ with notation as in Step 2 we can write $\varphi = \sum_{a \in \mathrm{E}_n} c_a \chi_{\Omega(a)}$ for some $n \in \mathbb{N}$ and with coefficients $c_a \geq 0$. Since Q is invertible by Proposition 5.12 we get for $F \in \mathrm{E}_\alpha^+$

$$\begin{aligned} (Q^{-1}F)(\varphi) &= (Q^{-1}F)(\sum_{a \in \mathrm{E}_n} c_a \chi_{\Omega(a)}) \\ &= e^{-(n-1)\alpha} \sum_{a \in E_n} c(a) F(a) \quad \geq \quad 0 \end{aligned}$$

and obtain $E_\alpha^+ \subset Q(\mathcal{D}_\alpha^+)$. □

5.15 Corollary. *For any subgroup $H < \mathrm{Is}\,(\mathcal{T})$, the map $Q : \mathcal{D}_\alpha \longrightarrow E_\alpha$ induces a bijection between positive α-dimensional densities for H and positive H-fixed functions in E_α.*

Proof. This follow directly from Corollary 5.14 and Corollary 5.13. □

5.5 Densities for the fundamental group

Positive α-dimensional densities for subgroups H of the isometry group are quite interesting, since they give raise to H-invariant measures for the dynamical system $(\mathcal{RT}, L_{\mathcal{R}})$, which are also invariant under the $\mathbb{Z}$-action of $L_{\mathcal{R}}$ (see [8], Section 6).

In this section we can identify these densities, using their correspondence to eigenvectors of a finite matrix through Corollary 5.15. This matrix looks very similar to the Markov matrix introduced in the first section of this chapter. With notation as there we take $\mathcal{T}$ to be the universal cover of an edge-indexed graph (A, i_A) with fundamental group $G < \mathrm{Is}\,(\mathcal{T})$ and projection $\pi : \mathcal{T} \to G\backslash\mathcal{T} = A$ to the finite connected graph A.

We are interested in all functions $F \in \mathbb{C}(\mathrm{E}\mathcal{T})$, that are solutions to the equations

$$\begin{aligned} h * F &= F \quad \text{for all} \quad h \in G \\ RF &= e^{\alpha} F \\ F(a) &\geq 0 \quad \text{for all} \quad a \in \mathrm{E}\mathcal{T}. \end{aligned} \tag{5.24}$$

We may start solving these equations beginning with the top one downwards.

Functions $F \in \mathbb{C}(\mathrm{E}\mathcal{T})$ which are invariant under G are precisely those assuming a constant values on each G-orbit on $\mathrm{E}\mathcal{T}$. So the first equation is solved if we look for functions in the subspace E^G of G-invariant functions. We will use the natural base $\{\chi_{\mathbf{a}} : \mathbf{a} = Ga \text{ for some edge } a \in \mathrm{E}\mathcal{T}\}$ for this vector space. A base element is defined for edges $a \in \mathrm{E}\mathcal{T}$ as

$$\chi_{\mathbf{a}}(a) = \begin{cases} 1 & \text{for } a \in \mathbf{a} \\ 0 & \text{for } a \notin \mathbf{a}. \end{cases}$$

Using the projection $\pi : \mathrm{E}\mathcal{T} \to \mathrm{E}A$ this base can be written without ordering as

$$\{\chi_{\mathbf{a}} : \mathbf{a} \in \mathrm{E}A\}.$$

The coordinates $[F]_{\mathbf{a}}$ ($\mathbf{a} \in \mathrm{E}A$) of the column vector $[F] \in \mathbb{C}^{|\mathrm{E}A|}$ representing F are fixed by

$$F = \sum_{\mathbf{a} \in \mathrm{E}A} [F]_{\mathbf{a}} \chi_{\mathbf{a}},$$

see Halmos [20] for notation of linear algebra. We will use functions

$$\delta_{\clubsuit,\diamondsuit} = \begin{cases} 1 & \text{if } \clubsuit = \diamondsuit \\ 0 & \text{if } \clubsuit \neq \diamondsuit \end{cases}$$

for whatever set the two elements $\clubsuit, \diamondsuit$ may belong to. Note for all functions F and edges $e \in \mathrm{E}\mathcal{T}$

$$F(e) = \sum_{\mathbf{a} \in \mathrm{E}A} [F]_{\mathbf{a}} \chi_{\mathbf{a}}(e) = \sum_{\mathbf{a} \in \mathrm{E}A} [F]_{\mathbf{a}} \delta_{\mathbf{a},\pi(e)} = [F]_{\pi e}, \tag{5.25}$$

in particular $\chi_{\mathbf{a}}(e) = [\chi_{\mathbf{a}}]_{\pi e} = \delta_{\mathbf{a},\pi e}$ for elements of the base.

As an approach to the second equation of (5.24) we are going to use the introduced base of E^G in order to calculate a matrix $[R]$ for R. Before doing so one must check, that R really induces an operator on the subspace $E^G \subset \mathbb{C}(\mathrm{E}\mathcal{T})$.

Observe that R is an $\mathrm{Is}(\mathcal{T})$-equivariant map, i.e. $h * R(F) = R(h * F)$ for all isometries h:

$$\begin{aligned}(h * R(F))(a) &= R(F)(h^{-1}a) = \sum_{\substack{\mathrm{o}(b)=\mathrm{t}(h^{-1}a)\\ b\neq\overline{h^{-1}a}}} F(b)\\ &= \sum_{\substack{\mathrm{o}(hb)=\mathrm{t}(a)\\ hb\neq\overline{a}}} F(b) = \sum_{\substack{\mathrm{o}(c)=\mathrm{t}(a)\\ c\neq\overline{a}}} F(h^{-1}c) = \sum_{\substack{\mathrm{o}(c)=\mathrm{t}(a)\\ c\neq\overline{a}}} (h * F)(c)\\ &= R(h * F)(a).\end{aligned} \tag{5.26}$$

One has now $RF(ga) = g^{-1} * (RF)(a) = R(g^{-1} * F)(a) = RF(a)$ for all $g \in G$ and $a \in \mathrm{E}\mathcal{T}$, if F is g-invariant. In this case, the function RF is also g-invariant, hence R is a linear map $E^G \to E^G$.

The entries of the matrix $[R]$ are fixed by the equations ($\mathbf{b} \in \mathrm{E}A$)

$$R\chi_{\mathbf{b}} = \sum_{\mathbf{a}\in\mathrm{E}A} [R]_{\mathbf{a},\mathbf{b}}\chi_{\mathbf{a}}. \tag{5.27}$$

To solve for the coefficients, we assume $\mathbf{b} \in \mathrm{E}A$, $e \in \mathrm{E}\mathcal{T}$.

$$R\chi_{\mathbf{b}}(e) = \sum_{\substack{\mathrm{o}(c)=\mathrm{t}(e)\\ c\neq\overline{e}}} \chi_{\mathbf{b}}(c) = \sum_{\mathrm{o}(c)=\mathrm{t}(e)} \chi_{\mathbf{b}}(c) - \chi_{\mathbf{b}}(\overline{e}).$$

For the last term holds $\chi_{\mathbf{b}}(\overline{e}) = \delta_{\mathbf{b},\pi\overline{e}} = \delta_{\overline{\mathbf{b}},\pi e} = \chi_{\overline{\mathbf{b}}}(e)$, the sum can be expanded as

$$\begin{aligned}\sum_{\mathrm{o}(c)=\mathrm{t}(e)} \chi_{\mathbf{b}}(c) &= \sum_{c\in\mathrm{St}^{\mathcal{T}}(\mathrm{t}(e))\cap\mathbf{b}} 1 \overset{(3.7)}{=} \delta_{\mathrm{o}(\mathbf{b}),\pi\mathrm{t}(e)}\mathrm{i}_A(\mathbf{b})\\ &= \sum_{\mathbf{a}\in\mathrm{E}A} \delta_{\mathbf{a},\pi e}\delta_{\mathrm{o}(\mathbf{b}),\mathrm{t}(\pi e)}\mathrm{i}_A(\mathbf{b}) = \sum_{\mathbf{a}\in\mathrm{E}A} \delta_{\mathbf{a},\pi e}\delta_{\mathrm{o}(\mathbf{b}),\mathrm{t}(\mathbf{a})}\mathrm{i}_A(\mathbf{b})\\ &= \sum_{\mathrm{o}(\mathbf{b})=\mathrm{t}(\mathbf{a})} \mathrm{i}_A(\mathbf{b})\delta_{\mathbf{a},\pi e} = \sum_{\mathrm{o}(\mathbf{b})=\mathrm{t}(\mathbf{a})} \mathrm{i}_A(\mathbf{b})\chi_{\mathbf{a}}(e).\end{aligned}$$

Equation (5.27) now becomes

$$R\chi_{\mathbf{b}}(e) = \sum_{\substack{\mathrm{o}(\mathbf{b})=\mathrm{t}(\mathbf{a})\\ \mathbf{b}\neq\overline{\mathbf{a}}}} \mathrm{i}_A(\mathbf{b})\chi_{\mathbf{a}}(e) + (\mathrm{i}_A(\mathbf{b}) - 1)\chi_{\overline{\mathbf{b}}}(e)$$

for all $e \in \mathrm{E}\mathcal{T}$, the matrix coefficients can be extracted as

$$[R]_{\mathbf{a},\mathbf{b}} = \begin{cases} \mathrm{i}_A(\mathbf{b}) & \text{if } \mathrm{o}(\mathbf{b}) = \mathrm{t}(\mathbf{a}) \text{ and } \mathbf{b} \neq \overline{\mathbf{a}}\\ \mathrm{i}_A(\mathbf{b}) - 1 & \text{if } \mathbf{b} = \overline{\mathbf{a}}\\ 0 & \text{otherwise.} \end{cases} \tag{5.28}$$

We call a column vector *positive*, if all its components are greater equal zero, *strictly positive*, if all its components are greater than zero. Note that the equation $RF = \lambda F$ is equivalent to the corresponding equation $[R][F] = \lambda[F]$ in coordinates. By equation (5.25) the function F is positive, i.e. $F(a) \geq 0$ for all edges $a \in \mathrm{E}\mathcal{T}$ if and only if $[F]$ is positive. Therefore problem (5.24) is equivalent with the problem of finding positive vectors $[F] \in \mathbb{C}^{|\mathrm{E}A|}$ such that

$$[R][F] = \lambda[F] \tag{5.29}$$

holds for some $\lambda > 0$. In many cases a solution to this problem comes form a Theorem by Perron-Frobenius:

5.16 Proposition. *If (A, i_A) is a finite irreducible edge indexed graph, then $[R]$ has a unique strictly positive eigenvector $[F]$ (up to scaling). The corresponding eigenvalue is positive and there are no other positive eigenvectors.*

Proof. A direct comparison shows $[R]_{a,b} \geq M_{a,b}$ for all edges a, b. M is the Markov matrix introduced at (5.2). This is also true for higher powers, since for two matrices B, C with $0 \leq B_{a,b} \leq C_{a,b}$ one has inductively

$$\begin{aligned} (B^{n+1})_{a,b} &= \sum_c (B^n)_{a,c} B_{c,b} \\ \leq \sum_c (C^n)_{a,c} C_{c,b} &= (C^{n+1})_{a,b} \end{aligned} \tag{5.30}$$

for all edges $a, b \in \mathrm{E}A$. We can use now $([R]^n)_{a,b} \geq (M^n)_{a,b}$ for all $n \in \mathbb{N}$ and all $a, b \in \mathrm{E}A$.

Since the graph (A, i_A) is irreducible there is by for each pair of edges a, b in A a positive path of positive length $\mathrm{len}(a, b) \geq 1$ joining the vertices a and b in the Markov graph $\mathcal{L}^+(A, \mathrm{i}_A)$. Then by Lemma 1.9.4 in [3], the matrix entry $(M^{\mathrm{len}(a,b)})_{a,b}$ is positive (M^n counts the number of paths with length n). This shows $([R]^{\mathrm{len}(a,b)})_{a,b} \geq (M^{\mathrm{len}(a,b)})_{a,b} > 0$.

The definition in [21] for a real square matrix $[R]$ with non-negative entries to be *irreducible* is that for each pair of indices i, j there is a number $n \in \mathbb{N}$ ($n \geq 1$) such that $([R]^n)_{i,j} > 0$. This is exactly what we verified for the matrix $[R]$. Thus by Theorem 1 (Perron-Frobenius) in [21], the matrix $[R]$ has a unique strictly positive eigenvector $[F]$ (up to scaling). This vector is associated to an eigenvalue $\lambda > 0$. Moreover the vector $[F]$ is the only positive eigenvector of $[R]$. ☐

With a glance at the original problem of finding positive α-dimensional densities for subgroups of the isometry group, Proposition 5.16 has the following translation by Corollary 5.15:

If (A, i_A) is a finite irreducible edge-indexed graph, then there exists a unique positive α-dimensional density on the universal cover $\mathcal{T}$ for the fundamental group $G < \mathrm{Is}\,(\mathcal{T})$. Since Q^{-1} transforms functions to functionals which evaluate over indicator functions of spheres as

$$\lambda_x(\chi_{\Omega(a)}) = e^{-(\mathrm{d}(x,\mathrm{o}(a))\alpha}F(a)$$

(cf. (5.21)), these functionals take positive values on all functions of $S\,(\mathcal{T}(\infty))^+ \setminus \{0\}$, cf. (5.23) and (4.18).

Proposition 5.16 is quite potent. In view of Theorem 1 and Corollary 2.9, all graphs of the class (2.4) are irreducible except for these in $\mathcal{BG}$. Example 5.18 gives a generalizable treatment for finding all positive vectors to the problem $[R][F] = e^{\alpha}[F]$ for graphs in $\mathcal{BG}$.

Before writing some examples, it may be convenient to translate equation (5.29) entirely to the underling graph A. A base for the function space $\mathrm{E}A$ can be defined for edges $\mathbf{a}, \mathbf{b} \in \mathrm{E}A$ as

$$\delta_{\mathbf{a}}(\mathbf{b}) := \delta_{\mathbf{a},\mathbf{b}}.$$

The vector coefficients of a function $F \in \mathbb{C}(\mathrm{E}A)$ are given by

$$[F]_{\mathbf{a}} = \sum_{\mathbf{b}\in \mathrm{E}A} [F]_{\mathbf{b}}\delta_{\mathbf{b}}(\mathbf{a}) = F(\mathbf{a}). \tag{5.31}$$

We use an isomorphism $E^G \to \mathbb{C}(\mathrm{E}A)$ defined by $I(\chi_{\mathbf{a}}) := \delta_{\mathbf{a}}$ and its inverse J given by $J(\delta_{\mathbf{a}}) := \chi_{\mathbf{a}}$. The operator R^A, corresponding to R becomes $R^A = IRJ$.

$$\begin{array}{ccc} E^G & \xrightarrow{R} & E^G \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle I} \\ \mathbb{C}(\mathrm{E}A) & \xrightarrow[R^A]{} & \mathbb{C}(\mathrm{E}A) \end{array}$$

A simple direct calculation yields $[I]_{\mathbf{a},\mathbf{b}} = \delta_{\mathbf{a},\mathbf{b}}$ and $[J]_{\mathbf{a},\mathbf{b}} = \delta_{\mathbf{a},\mathbf{b}}$ for the matrix coefficients of I and J. Thus

$$\begin{aligned} [R^A]_{\mathbf{a},\mathbf{b}} &= ([I][R][J])_{\mathbf{a},\mathbf{b}} = \sum_{\mathbf{c},\mathbf{d}} [I]_{\mathbf{a},\mathbf{c}}[R]_{\mathbf{c},\mathbf{d}}[J]_{\mathbf{d},\mathbf{b}} \\ &= \sum_{\mathbf{c},\mathbf{d}} \delta_{\mathbf{a},\mathbf{c}}[R]_{\mathbf{c},\mathbf{d}}\delta_{\mathbf{d},\mathbf{b}} = [R]_{\mathbf{a},\mathbf{b}}. \end{aligned}$$

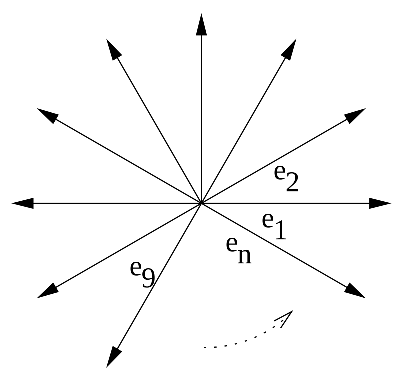
For this equality we write simply R for R^A in future appearances. In coordinate free notation

$$
\begin{aligned}
(RF)(\mathbf{a}) \quad &\overset{(5.31)}{=} \quad [RF]_{\mathbf{a}} \quad = \quad ([R][F])_{\mathbf{a}} \quad = \quad \sum_{\mathbf{b}\in \mathrm{E}A} [R]_{\mathbf{a},\mathbf{b}}[F]_{\mathbf{b}} \\
&\overset{(5.31)}{=} \quad \sum_{\mathbf{b}\in \mathrm{E}A} [R]_{\mathbf{a},\mathbf{b}} F(\mathbf{b}).
\end{aligned}
\tag{5.32}
$$

In what follows, we are going to change between writing vectors by coordinates or as functions where it seems most appropriate.

5.17 Example (Star graphs). A *star graph* is a tree represented ($n \geq 2$) by the diagram in Figure 5.4. The star graph as a tree is irreducible by Corollary 2.9 and Theorem 1. (Confer in Section 2.6 the last example.) Therefore by

Figure 5.4: A star graph



Proposition 5.16 the matrix $[R]$ has a unique strictly positive eigenvector $[F]$ to a positive eigenvalue λ.

A formula can be given for $[F]$ if there is $M \geq 2$ such that $\mathrm{i}(\overline{e_i}) = M$ for all i.[1] We make the heuristic assumptions $F(e_i) = 1$ and $F(\overline{e_i}) = s$ for all $i \in \{1, \ldots, n\}$. One has

$$RF(e_i) = (M-1)F(\overline{e_i}), \tag{5.33}$$

so $\lambda := (M-1)s$ gives

$$RF(e_i) = \lambda = \lambda F(e_i) \tag{5.34}$$

[1] If the indexing is not constant on the edges $\overline{e_i}$, a calculation of $[F]$ is more complicated. See Example 5.23.

as well as

$$RF(\overline{e_i}) \overset{(5.33)}{=} \frac{1}{M-1} R^2 F(e_i) \overset{(5.34)}{=} \frac{\lambda^2}{M-1} = \lambda s = \lambda F(\overline{e_i}). \tag{5.35}$$

This shows that the assumptions have been well chosen. λ and s can actually be calculated as follows.

$$\lambda^2 \overset{(5.35)}{=} (M-1) RF(\overline{e_1}) = (M-1) \left((\mathrm{i}(e_1) - 1)\, F(e_1) + \sum_{j=2}^{n} \mathrm{i}(e_j) F(e_j) \right)$$
$$= (M-1) \left(\sum_{j=1}^{n} \mathrm{i}(e_j) - 1 \right),$$

whence for

$$\lambda = \sqrt{(M-1) \left(\sum_{j=1}^{n} \mathrm{i}(e_j) - 1 \right)} \quad \text{and} \quad s = \frac{\lambda}{M-1}, \tag{5.36}$$

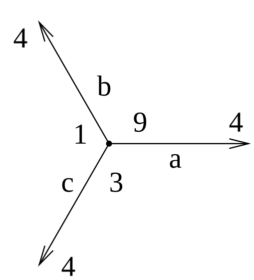
the unique positive eigenvector of $[R]$ is given by

$$[F] = (\underbrace{\underbrace{1, s}, \ldots, \underbrace{1, s}}_{n \text{ times}})^{\mathrm{T}}.$$

We used the ordering $(e_1, \overline{e_1}, \ldots, e_n, \overline{e_n})$ for the components of the column vector.

These results shall be applied to the graph in Figure 5.5. Then $\lambda = \sqrt{3 \cdot 12} = 6$ and $s = \frac{\lambda}{3} = 2$. Indeed, (with ordering $a, \overline{a}, b, \overline{b}, c, \overline{c}$ of vector components) the

Figure 5.5: A special indexing for a star graph



matrix of R is given by

$$[R] = \begin{pmatrix} 0 & 3 & 0 & 0 & 0 & 0 \\ 8 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 9 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 9 & 0 & 1 & 0 & 2 & 0 \end{pmatrix}$$

and $[F] = \Big(1,2,1,2,1,2\Big)^{\mathrm{T}}$ satisfies $[R][F] = 6[F]$.

5.18 Example (Non-irreducible Circuits — graphs of $\mathcal{BG}$). We solve $[R][F] = e^{\alpha}[F]$ for circ_4 as far as possible in two settings; the uni-modular indexing $(\mathrm{circ}_4, 1)$, and a non-unimodular indexing. The methods may be extended to general non-irreducible circuits. The base elements of the space of $\mathbb{C}(\mathrm{Ecirc}_4)$ shall be ordered by $(a, b, c, d, \overline{a}, \overline{b}, \overline{c}, \overline{d})$. The edges are labeled in the diagram in Figure 5.6.

The matrix of R for an indexing constant to one is given by

$$[R] = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & D \end{array}\right) := \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & & & & \\ 0 & 0 & 1 & 0 & & 0 & & \\ 0 & 0 & 0 & 1 & & & & \\ 1 & 0 & 0 & 0 & & & & \\ \hline & & & & 0 & 0 & 0 & 1 \\ & 0 & & & 1 & 0 & 0 & 0 \\ & & & & 0 & 1 & 0 & 0 \\ & & & & 0 & 0 & 1 & 0 \end{array}\right).$$

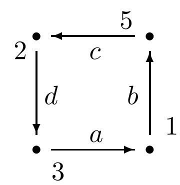
We split up the equation $[R][F] = \lambda[F]$ into the two independent equations

$$\begin{aligned} BV &= \lambda V \\ DW &= \lambda W \end{aligned}$$

for $V = ([F]_a, [F]_b, [F]_c, [F]_d)^{\mathrm{T}}$ and $W = ([F]_{\overline{a}}, [F]_{\overline{b}}, [F]_{\overline{c}}, [F]_{\overline{d}})^{\mathrm{T}}$. The action of B on another matrix is a cyclic permutation of rows respectively columns. Therefore B is irreducible. The eigenvector $(1,1,1,1)^{\mathrm{T}}$ is therefore by Perron-Frobenius the only strictly positive eigenvector. One can show the same property for D. The positive eigenvectors of $[R]$ form thus the set of $\mathbb{R}$-dimension two $\{(c_1, c_1, c_1, c_1, c_2, c_2, c_2, c_2)^{\mathrm{T}} : c_1, c_2 \geq 0\}$ to the common eigenvector 1.

In case that one orientation has indices greater than one (Figure 5.6), there is a different situation. Under the same base as above, the matrix of R becomes

Figure 5.6: A non-irreducible edge-indexed graph



$$[R] = \left(\begin{array}{c|c} B & 0 \\ \hline C & D \end{array}\right) := \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & & & & \\ 0 & 0 & 5 & 0 & & & 0 & \\ 0 & 0 & 0 & 2 & & & & \\ 3 & 0 & 0 & 0 & & & & \\ \hline 2 & & & & 0 & 0 & 0 & 1 \\ & 0 & & & 1 & 0 & 0 & 0 \\ & & 4 & & 0 & 1 & 0 & 0 \\ & & & 1 & 0 & 0 & 1 & 0 \end{array}\right).$$

We split up the equation $[R][F] = \lambda[F]$ into the two equations

$$\begin{aligned} BV &= \lambda V \\ CV + DW &= \lambda W \end{aligned}$$

If one chooses $V = 0$, then as above the resulting eigenvectors of $[R]$ to the eigenvalue 1 are of the form $\{(0, 0, 0, 0, c, c, c, c)^{\mathrm{T}} : c \geq 0\}$.

But there are also strictly positive eigenvectors of $[R]$. By (5.30) the matrix B is irreducible, hence has a unique strictly positive eigenvector V to a positive eigenvalue $\lambda > 0$. We can actually calculate the value of λ. Multiplication of block matrices gives

$$[R]^n = \begin{pmatrix} B^n & 0 \\ * & * \end{pmatrix}$$

Thus

$$\begin{aligned} (B^4V)_a &= ([R]^4[F])_a = R^4F(a) = R^3F(b) = 5R^2F(c) \\ &= 10RF(d) = 30F(a) = 30[F]_a = 30V_a \end{aligned}$$

for any vector V. On the other hand

$$(B^4V)_a = \lambda^4 V_a$$

gives $\lambda = \sqrt[4]{30} \approx 2,3403 > 1$, since $\lambda, V_a > 0$. If we plug this solution into the second equation, we have to solve for W in

$$(\lambda\mathbb{I} - D)W = CV.$$

Now $(\lambda\mathbb{I} - D)(D^3 + \lambda D^2 + \lambda^2 D + \lambda^3\mathbb{I}) = (\lambda^4\mathbb{I} - D^4) = (\lambda^4 - 1)\mathbb{I}$, since $D^4 = \mathbb{I}$. As $\lambda > 1$

$$(\lambda\mathbb{I} - D)^{-1} = \frac{1}{\lambda^4 - 1}(D^3 + \lambda D^2 + \lambda^2 D + \lambda^3\mathbb{I})$$

and this matrix has purely non-negative entries, in fact only positive entries. As V is strictly positive and since C has all coefficients non-negative and some positive, $W = (\lambda\mathbb{I} - D)^{-1}CV$ is strictly positive.

Unlike the case of indexing one, the strictly positive eigenvectors of $[R]$ form here a set of $\mathbb{R}$-dimension one like in the case of irreducible graphs. But in contrast to these graphs, we get here also a positive eigenvector, which is not strictly positive, a behavior which has been found for circuits with indexing one only.

For an explicit example we calculate the first graph in Figure 2.4. The matrix $[R]$ is given by

$$\left(\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 2 & 0 & & \\ \hline 1 & & 0 & 1 \\ & 0 & 1 & 0 \end{array}\right)$$

and has the eigenvector $(0,0,1,1)^{\mathrm{T}}$ to the eigenvalue 1 and the eigenvector $(1, \sqrt{2}, \sqrt{2}, 1)^{\mathrm{T}}$ to the eigenvalue $\sqrt{2}$.

5.6 Densities as Markov measures

The main derivations that have been done so far, will come together in this section. We consider a finite, connected and unimodular edge-indexed graph (A, i_A). We take G as the fundamental group, $\mathcal{T}$ the universal cover, G_f the full group and π the projection morphism. The geometric construction of the border of a tree and the identified α-dimensional densities as eigenfunctions of the matrix $[R]$ shall be used to write a Markov measure for the topological Markov chain defined over the oriented line graph $\mathcal{L}^+(A, \mathrm{i}_A)$.

We follow Section 7.3 in [8] from M.Burger and S.Mozes. Prior and more generally, in Section 6.2, the authors use positive α-dimensional densities for G_f to define $\mathbb{Z}$-and G_f-invariant measures on $\mathcal{R}(\mathcal{T})$, which then define Z-invariant measures on $\mathcal{G}(A, \mathrm{i}_A)$ via the projection from the tree $\mathcal{T}$ to the graph A, provided that G_f is a closed and uni-modular subgroup of the isometry group. They use the description of reduced paths in terms of two border points of $\mathcal{T}(\infty)$ and an integer (see Section 5.3).

We are not going to write these abstract arguments here, instead we show how α-dimensional densities for G_f can be used to define Markov measures directly for the topological Markov chain $(\mathcal{P}(\mathcal{L}^+(A, \mathrm{i}_A)), L_{\mathcal{P}})$. Thus continuing we too use a Haar measure on G_f and must assume G_f to be unimodular in order to have sufficiently many equations.

These assumptions are justified, since G_f is a locally compact topological group by Corollary 3.16. Hence there exists a (left-invariant) Haar measure on the Borel algebra of G_f by Theorem B, Sec. 58, [22]. By Proposition 3.15 we know that G_f is a closed subgroup of $\mathrm{Is}\,(\mathcal{T})$. Further G_f acts without inversions on $\mathcal{T}$. H.Bass and R.Kulkarni prove with this in [11], Corollary 3.7, that G_f is unimodular if and only if the edge-indexed graph (A, i_A) is unimodular.

As first step, we take advantage of unimodularity. Suppose that m is a left- and right-invariant Haar measure on the group G_f. For each natural number $n \geq 2$ and a geodesic with edge sequence $e_1, \ldots, e_n$ in (A, i_A) we can choose by Lemma 5.2 a lift $p = \tilde{e}_1, \ldots, \tilde{e}_n$ as reduced path in $\mathcal{T}$. We take K_p to be the stabilizer of p in G_f.

$$K_p := (G_f)_p = \bigcap_{x \in \{\mathrm{o}(\tilde{e}_1), \ldots, \mathrm{o}(\tilde{e}_n), \mathrm{t}(\tilde{e}_n)\}} (G_f)_x.$$

By Corollary 3.16 K_p is an open and compact subgroup of G_f, hence as non-empty set has positive and finite measure

$$0 < m(K_p) < \infty.$$

Observe that for any other lifted reduced path q of $e_1, \ldots, e_n$ there is $g \in G_f$

such that $q = gp$ (Lemma 5.4). It follows from

$$\begin{aligned} K_{gp} &= \{h \in G_f : hgp = gp\} &= \{h \in G_f : g^{-1}hgp = p\} \\ &= \{gkg^{-1} \in G_f : kp = p\} &= g(K_p)g^{-1} \end{aligned}$$

by uni-modularity of G_f that $m(K_q) = m(gK_pg^{-1}) = m(K_p)$. Thus $m(K_p)$ only depends on the geodesic $e_1, \ldots, e_n$ and we can define

$$m(e_1, \ldots, e_n) := m(K_{\tilde{e}_1, \ldots, \tilde{e}_n}). \tag{5.37}$$

The stabilizer $K_{\tilde{e}_{n-1}}$ acts on the set $\mathrm{St}^{\mathcal{T}}_{\mathrm{o}(\tilde{e}_n)} = \{e \in \mathrm{E}\mathcal{T} : \mathrm{o}(e) = \mathrm{o}(\tilde{e}_n)\}$ since $\mathrm{o}(\tilde{e}_n)$ is fixed by all these isometries. The orbit of $\tilde{e}_n$ under this action is given by

$$K_{\tilde{e}_{n-1}}(\tilde{e}_n) = \pi^{-1}_{\mathrm{o}(\tilde{e}_n)}(e_n) \setminus \{\overline{\tilde{e}_{n-1}}\}.$$

The map $\pi_{\mathrm{o}(\tilde{e}_n)}$ is the local projection $\mathrm{E}\mathcal{T} \to \mathrm{E}A$ from the star at $\mathrm{o}(\tilde{e}_n)$ to the star at $\mathrm{o}(e_n)$. To verify this equation, observe that $\overline{\tilde{e}_{n-1}}$ is fixed by all isometries of $K_{\tilde{e}_{n-1}}$, thus showing $\overline{\tilde{e}_{n-1}} \notin K_{\tilde{e}_{n-1}}(\tilde{e}_n)$ because $\tilde{e}_{n-1}, \tilde{e}_n$ is reduced. On the other hand, if $a \in \pi^{-1}_{\mathrm{o}(\tilde{e}_n)}(e_n) \setminus \{\overline{\tilde{e}_{n-1}}\}$, one can show $a = g(\tilde{e}_n)$ for some $g \in K_{\tilde{e}_{n-1}}$. By Lemma 3.11 there is an isometry $g \in G_f$ with $a = g\tilde{e}_n$ and g is the identity on the connected component $\mathcal{C}$ at $\mathrm{o}(\tilde{e}_n)$ in the graph $\mathcal{T} \setminus \{\tilde{e}_n, \overline{\tilde{e}_n}, a, \overline{a}\}$. Since $\tilde{e}_1, \ldots, \tilde{e}_{n-1}$ is reduced, and since $\overline{\tilde{e}_{n-1}} \notin \{a, \tilde{e}_n\}$, the whole path $\tilde{e}_1, \ldots, \tilde{e}_{n-1}$ is in $\mathcal{C}$, hence fixed by g. This shows $g \in K_{\tilde{e}_{n-1}}$. Thus by equation (3.7)

$$|K_{\tilde{e}_{n-1}}(\tilde{e}_n)| = \begin{cases} \mathrm{i}_A(e_n) & \text{for } e_n \neq \overline{e_{n-1}} \\ \mathrm{i}_A(e_n) - 1 & \text{for } e_n = \overline{e_{n-1}}. \end{cases}$$

The stabilizer of $\tilde{e}_n$ in $K_{\tilde{e}_{n-1}}$ is given by $K_{\tilde{e}_{n-1}, \tilde{e}_n}$, thus Proposition I, 5.1 in [17] shows

$$[K_{\tilde{e}_{n-1}} : K_{\tilde{e}_{n-1}, \tilde{e}_n}] = |K_{\tilde{e}_{n-1}}(\tilde{e}_n)|.$$

The cosets of a group form a partition of the same group, thus one has by additivity and invariance of the Haar measure $|K_{\tilde{e}_{n-1}}(\tilde{e}_n)| \cdot m(K_{\tilde{e}_{n-1}, \tilde{e}_n}) = m(K_{\tilde{e}_{n-1}})$, in other words

$$\frac{m(e_{n-1})}{m(e_{n-1}, e_n)} = \begin{cases} \mathrm{i}_A(e_n) & \text{for } e_n \neq \overline{e_{n-1}} \\ \mathrm{i}_A(e_n) - 1 & \text{for } e_n = \overline{e_{n-1}}. \end{cases} \tag{5.38}$$

This formula can easily be extended by exactly the same arguments to the more general case with the statement $(1 \le k \le n-1)$

$$\frac{m(e_k,\ldots,e_{n-1})}{m(e_k,\ldots,e_n)} = \begin{cases} \mathrm{i}_A(e_n) & \text{for } e_n \neq \overline{e_{n-1}} \\ \mathrm{i}_A(e_n)-1 & \text{for } e_n = \overline{e_{n-1}}. \end{cases} \tag{5.39}$$

We copy from [21] all that is necessary for the introduction of Markov measures. The Markov matrix M given by

$$M_{a,b} = \begin{cases} 1 & \text{if } a,b \text{ is a geodesic in } (A,\mathrm{i}_A) \\ 0 & \text{if } a,b \text{ is not a geodesic in } (A,\mathrm{i}_A) \end{cases} \tag{5.40}$$

describes exactly the space of positive paths in $\mathcal{P}(\mathcal{L}^+(A,\mathrm{i}_A))$ by sequences of vertices as

$$\mathcal{P}(\mathcal{L}^+(A,\mathrm{i}_A)) = \left\{ w \in \mathrm{V}\mathcal{L}^+(A,\mathrm{i}_A)^{\mathbb{Z}} : M_{w(i),w(i+1)} = 1 \quad \text{for all } i \in \mathbb{Z} \right\}.$$

Lemma 2.2 shows that such a sequence defines a positive path. For any positive path with vertex sequence $e_1,\ldots,e_n$ and any integer j a *cylinder* is defined as

$$Z(j,e_1,\ldots,e_n) = \left\{ w \in \mathcal{P}(\mathcal{L}^+(A,\mathrm{i}_A)) : w(j+i) = e_i \quad \text{for all } 1 \le i \le n \right\}.$$

These cylinders form the base of a topology for $\mathcal{P}(\mathcal{L}^+(A,\mathrm{i}_A))$. A matrix $P \in \mathbb{C}^{|\mathrm{E}A|\times|\mathrm{E}A|}$ is called a *stochastic matrix* if all its entries are not negative and $\sum_{b\in \mathrm{E}A} P_{a,b} = 1$ for all $a \in \mathrm{E}A$. A *stationary probability vector* for the stochastic matrix P is a positive left eigenvector p to the stochastic matrix P, i.e. all entries are not negative and $\sum_{a\in\mathrm{E}A} p_a P_{a,b} = p_b$ for all $b \in \mathrm{E}A$. If we suppose, that $P_{a,b} = 0$ if $M_{a,b} = 0$ the *Markov measure* $\mu_{P,p}$ is be defined on cylinders as

$$\begin{aligned} \mu_{P,p}(Z(j,e_1)) &= p_{e_1} \quad \text{and} \\ \mu_{P,p}(Z(j,e_1,\ldots,e_n)) &= p_{e_1}P_{e_1,e_2}\cdots P_{e_{n-1},e_n} \quad \text{for } n \ge 2. \end{aligned} \tag{5.41}$$

This definition extends consistently to the Borel algebra of $\mathcal{P}(\mathcal{L}^+(A,\mathrm{i}_A))$ and $\mu_{P,p}$ is invariant under the shift operator $L_{\mathcal{P}}$.

5.19 Proposition. *Suppose F is a strictly positive eigenvector to the matrix $[R]$ defined at (5.28) with eigenvalue e^{α}. Then the matrix P with coefficients*

$$P_{a,b} = e^{-\alpha}\frac{F(b)}{F(a)}[R]_{a,b}$$

for all edges $a,b \in EA$ is a stochastic matrix.

Proof.

$$\begin{aligned}\sum_{b\in \mathrm{E}A} P_{a,b} &= \sum_{b\in \mathrm{E}A} e^{-\alpha}\tfrac{F(b)}{F(a)}[R]_{a,b} = \tfrac{e^{-\alpha}}{F(a)}\sum_{b\in \mathrm{E}A}[R]_{a,b}F(b)\\ &= \tfrac{e^{-\alpha}}{F(a)}e^{\alpha}F(a) = 1.\end{aligned}$$

□

Since $M_{a,b} = 0$ implies by definition $[R]_{a,b} = 0$, we see that in this case also $P_{a,b} = 0$, so P would define a Markov measure. We would like to calculate a stationary probability vector. We want even more, we ought to have a stationary probability vector in terms of the tree $\mathcal{T}$ and the group G_f only. Thus we assume that the graph (A, i_A) is uni-modular. Then definition (5.37) can be set.

5.20 Proposition. *If (A, i_A) is a finite connected and uni-modular edge indexed graph and F is a strictly positive eigenvector to the matrix $[R]$ defined at (5.28) with eigenvalue e^{α}, then the vector p defined in coordinates as*

$$p_a = \frac{F(\overline{a})F(a)}{m(a)} \tag{5.42}$$

for all edges $a \in \mathrm{E}A$ is a stationary probability vector (up to scaling) for the stochastic matrix P defined in Proposition 5.19.

Proof. If a, b is a geodesic in (A, i_A), then by definition and equation (5.38) one has $[R]_{a,b} = \frac{m(a)}{m(a,b)}$. The stabilizer of the path a is equal to the stabilizer of $\overline{a}$. The stabilizer of a, b equals the one $\overline{b}, \overline{a}$, hence $\frac{m(b)}{m(a,b)} = \frac{m(\overline{b})}{m(\overline{b},\overline{a})} = [R]_{\overline{b},\overline{a}}$. If two edge a, b do not form a geodesic in this ordering, then $[R]_{a,b} = [R]_{\overline{b},\overline{a}} = 0$. Altogether

$$[R]_{a,b} = \frac{m(a)}{m(b)}[R]_{\overline{b},\overline{a}} \tag{5.43}$$

for all $a, b \in \mathrm{E}A$. Thus prepared we get

$$\begin{aligned}\sum_{a\in \mathrm{E}A} p_a P_{a,b} &= \sum_{a\in \mathrm{E}A} \tfrac{F(\overline{a})F(a)}{m(a)} e^{-\alpha}\tfrac{F(b)}{F(a)}[R]_{a,b}\\ &\overset{(5.43)}{=} \sum_{a\in \mathrm{E}A} \tfrac{F(\overline{a})F(a)}{m(a)} e^{-\alpha}\tfrac{F(b)}{F(a)}\tfrac{m(a)}{m(b)}[R]_{\overline{b},\overline{a}} = e^{-\alpha}\tfrac{F(b)}{m(b)}\sum_{a\in \mathrm{E}A}[R]_{\overline{b},\overline{a}}F(\overline{a})\\ &\overset{(+)}{=} e^{-\alpha}\tfrac{F(b)}{m(b)}\sum_{\overline{a}\in \mathrm{E}A}[R]_{\overline{b},\overline{a}}F(\overline{a}) = e^{-\alpha}\tfrac{F(b)}{m(b)}e^{\alpha}F(\overline{b}) = p_b.\end{aligned}$$

The step (+) is correct, since it does not matter if we sum over a or if we reorder this finite sum by summing over $\overline{a}$. □

Under the assumptions of Proposition 5.20 the measure of cylinders can be written more explicitly: For $n \geq 2$ and every positive path $e_1, \dots, e_n$ of the oriented line graph, formula (5.39) gives $[R]_{e_{n-1},e_n} = \frac{m(e_1,\dots,e_{n-1})}{m(e_1,\dots,e_n)}$. Omitting the integer j in the definition of cylinders (i.e. using $L_{\mathcal{P}}$-invariance of the measure) one gets inductively

$$\begin{aligned}
\mu_{P,p}(Z(e_1,\dots,e_n)) &= p_{e_1} P_{e_1,e_2} \cdots P_{e_{n-1},e_n} \\
&= \mu_{P,p}(Z(e_1,\dots,e_{n-1})) P_{e_{n-1},e_n} \\
&= \frac{F(\overline{e_1})F(e_{n-1})e^{-\alpha(n-2)}}{m(e_1,\dots,e_{n-1})} e^{-\alpha} \frac{F(e_n)}{F(e_{n-1})} [R]_{e_{n-1},e_n} \\
&= \frac{F(\overline{e_1})F(e_n)e^{-\alpha(n-1)}}{m(e_1,\dots,e_{n-1})} \frac{m(e_1,\dots,e_{n-1})}{m(e_1,\dots,e_n)} = \frac{F(\overline{e_1})F(e_n)e^{-\alpha(n-1)}}{m(e_1,\dots,e_n)}.
\end{aligned}$$

The root of induction is equation 5.42. This shows for all $n \geq 1$

$$\mu_{P,p}(Z(e_1,\dots,e_n)) = \frac{F(\overline{e_1})F(e_n)e^{-\alpha(n-1)}}{m(e_1,\dots,e_n)}. \tag{5.44}$$

This formula not only is more explicit, yet it exhibits *time-reversal symmetry* for cylinders. By σ-additivity, this property extends to arbitrary Borel sets.

5.21 Proposition. *With assumptions of Proposition 5.20 one has for all positive paths $e_1, \dots, e_n$ of length greater equal zero in $\mathcal{L}^+(A, \mathrm{i}_A)$*

$$\mu_{P,p}(Z(e_1,\dots,e_n)) = \mu_{P,p}(Z(\overline{e_n},\dots,\overline{e_1})).$$

Proof. Since m is defined by the measure of a stabilizer one gets $m(\overline{e_n},\dots,\overline{e_1}) = m(e_1,\dots,e_n)$. The claimed property follows then directly with equation 5.44. □

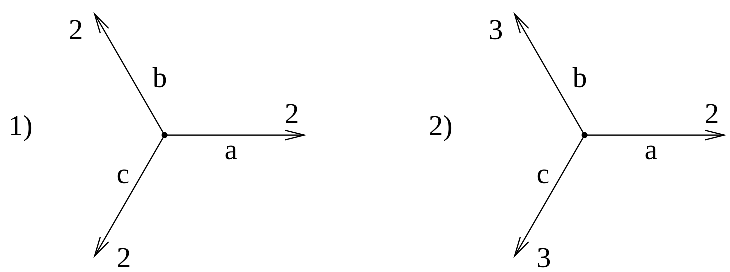
For an interpretation one should look at the edge-indexed graph (A, i_A). There the two geodesics $e_1, \dots, e_n$ and $\overline{e_n} \dots \overline{e_1}$ "traverse" the same geometrical edges in opposite direction. In the oriented line graph, the corresponding positive paths might have no vertices in common.

5.22 Example (No time-reversal symmetry). It seems an interesting question, whether or not all Markov measures defined as in Proposition 5.19 share this symmetry. An example of an irreducible graph, which is not unimodular and breaks with the symmetry can be given. We take circ_2 with the indexing $\mathrm{i}(\overline{[0,1]}) = 2$, $\mathrm{i}([1,0]) = 3$ and one elsewhere. With vector coefficients ordered as $[0,1], \overline{[0,1]}, [1,0], \overline{[1,0]}$ one has as stationary probability vector $p = \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}\right)^{\mathrm{T}}$, so that not even $\mu_{P,p}(Z([0,1])) = \mu_{P,p}(Z(\overline{[0,1]}))$.

However for a Markov matrix M there may be a lot (or none) of stochastic matrices vanishing where M vanishes. To find out more about the construction of Markov measures here, two examples shall illustrate that different indexings of the same graph can produce different Markov measures (Example 5.23) or they may also yield the same measure (Example 5.24).

5.23 Example. A positive eigenvector for the matrix $[R]$ to graph 1) in Figure 5.7 has been calculated in Example 5.17. The stochastic matrix P from

Figure 5.7: Two different star graphs with the same oriented line graph



Proposition 5.19 can be calculated as (the base is supposed be ordered like $a, \overline{a}, b, \overline{b}, c, \overline{c}$)

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}.$$

Since this matrix is irreducible, it is easily seen that $\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)^{\mathrm{T}}$ is the unique stationary probability vector for P.

For graph 2), the heuristic assumptions (from symmetry) $F(b) = F(c)$ and $F(\overline{b}) = F(\overline{c})$ lead to the positive eigenvector $F = \left(4, 4\lambda, 2\lambda^2, \lambda^3, 2\lambda^2, \lambda^3\right)^{\mathrm{T}}$ to the matrix $[R]$ with eigenvalue $\lambda = \sqrt{1+\sqrt{5}}$. From there one obtains

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \sigma^2 & 0 & 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \sigma^2 & 0 & \sigma & 0 & 0 & 0 \end{pmatrix}$$

with $\sigma = \frac{2}{\lambda^2} = \frac{-1+\sqrt{5}}{2}$ the *golden section*. The stationary probability vector for this matrix can be written as

$$p = \frac{1}{20}(2-\sigma)\Big(2, 2, 2+\sigma, 2+\sigma, 2+\sigma, 2+\sigma\Big)^{\mathrm{T}},$$

which is about $\Big(0.138, 0.138, 0.181, 0.181, 0.181, 0.181\Big)^{\mathrm{T}}$.

5.24 Example. Star graphs have been defined in Example 5.17. If we assume that $\mathrm{i}(e_i) = 1$ and $\mathrm{i}(\overline{e_i}) = M \geq 2$ for all $i \in \{1, \dots, n\}$, then

$$\Big(1, s, \dots, 1, s\Big)^{\mathrm{T}}$$

is a strictly positive eigenvector for $[R]$, with eigenvalue $\lambda = \sqrt{(M-1)(n-1)}$ and $s = \frac{\lambda}{M-1}$. (See equation (5.36).) From Proposition 5.19 we obtain, that the corresponding stochastic matrix is defined by

$$\begin{aligned} P_{e_i,e_j} &= P_{\overline{e_i},\overline{e_j}} = 0 \\ P_{e_i,\overline{e_j}} &= \tfrac{1}{\lambda}\tfrac{s}{1}\delta_{i,j}(M-1) = \delta_{i,j} \\ P_{\overline{e_i},e_j} &= \tfrac{1}{\lambda}\tfrac{1}{s}(1-\delta_{i,j}) = (1-\delta_{i,j})\tfrac{M-1}{\lambda^2} = (1-\delta_{i,j})\tfrac{1}{n-1} \end{aligned}$$

for all i, j. For a fixed $n \geq 2$, the graphs produce the same stochastic matrix for all M, hence the same Markov measure. The stationary probability vector will be calculated in Example 5.27.

An indexing i_A of a graph A is called *minimal*, if $\mathrm{i}_A(e) = 2$ on edges e whose origin is of degree one and $\mathrm{i}_A(e) = 1$ else. Minimality is a special case of unimodularity. Edges with a border vertex of degree one are not part of any closed reduced path, so by Lemma 2.4 the graph is unimodular. A special feature of graphs with minimal indexing is

$$[R]_{a,b} = [R]_{\overline{b},\overline{a}} \tag{5.45}$$

for all edges $a, b \in \mathrm{E}A$.

For every transitive Markov matrix M there is a distinguished Markov measure called the *Parry measure*, the construction of which is unique if M is irreducible. It has a special meaning, since (for transitive Markov matrices) it is the unique *measure of maximal entropy* for the dynamical system (cf. [3], Corollary 20.1.5). This measure also has an interpretation as the *asymptotic distribution of periodic orbits* (confer [3], page 177).

The Perry measure is defined as follows [21]. If v is a strictly positive (normalized, right) eigenvector of an irreducible Markov matrix M with eigenvalue $\lambda > 0$ then

$$P_{a,b} := M_{a,b} \frac{v(b)}{\lambda v(a)}$$

defines a stochastic matrix. The corresponding Markov measure is called *Parry measure.*

5.25 Proposition. *If* (A, i_A) *is a finite connected irreducible edge indexed graph with minimal edge indexing, then the stochastic matrix of Proposition 5.19 defines the Parry measure.*

Proof. Since i_A is minimal we get $[R] = M$, hence F is an eigenvector also to M with eigenvalue e^{α} and we can write

$$P_{a,b} = e^{-\alpha} \frac{F(b)}{F(a)} [R]_{a,b} = M_{a,b} \frac{F(b)}{e^{\alpha} F(a)}.$$

□

There is another convenient property of minimal indexings. One is saved from solving for the stationary probability vector p of P (the value of the Haar measure $m(e)$ is not known in general). This gives a slight reduction of calculations.

5.26 Proposition. *If* (A, i_A) *is a finite connected irreducible edge indexed graph with minimal edge indexing, then the probability vector for the stochastic matrix defined in Proposition 5.19 is given (up to scaling) by*

$$p_a := F(\overline{a}) F(a),$$

where F *is the strictly positive eigenvector of* $[R]$ *to the eigenvalue* e^{α}*.*

Proof. Since (A, i_A) has minimal indexing, the stochastic matrix P defines the Perry measure by the previous Proposition. The corresponding stationary probability vector is given [21] (up to scaling) by

$$p_a := u_a v_a,$$

where u is a left eigenvector and v is right eigenvector of the Markov matrix M.

One has $M = [R]$. Suppose F is a strictly positive right eigenvector of M, then $\overline{F}$ defined by $\overline{F}(a) := F(\overline{a})$ for all edges $a \in \mathrm{E}A$ satisfies for all edges $e \in \mathrm{E}A$

$$\begin{aligned}
(\overline{F}^{\mathrm{T}} M)(e) &= \sum_{a \in \mathrm{E}A} \overline{F}(a)[R]_{a,e} \overset{(5.45)}{=} \sum_{a \in \mathrm{E}A} [R]_{\overline{e},\overline{a}} F(\overline{a}) \\
&= \sum_{\overline{a} \in \mathrm{E}A} [R]_{\overline{e},\overline{a}} F(\overline{a}) = RF(\overline{e}) = e^{\alpha} F(\overline{e}) = e^{\alpha} \overline{F}(e) \\
&= (\overline{F}^{\mathrm{T}} e^{\alpha})(e).
\end{aligned}$$

This equation shows that $\overline{F}$ is a left eigenvector of M, thus $p_a = \overline{F}(a) F(a) = F(\overline{a}) F(a)$ for all edges $a \in \mathrm{E}A$. □

5.27 Example (Star graphs with minimal edge indexing). (See Figure 5.4 for a diagram of the graph.) Star graphs with minimal indexing and $n \geq 2$ geometric edges ($2n$ edges) have Parry measure and stationary probability vector given by

$$\left(\frac{1}{2n}, \dots, \frac{1}{2n}\right)^{\mathrm{T}}$$

(Proposition 5.26). Example 5.17 shows a calculation of the positive eigenvector $[F]$ to the matrix $[R]$.

5.7 Ergodic properties

The connection properties of a graph (A, i_A) will be related to the ergodic properties of the topological Markov chain $(\mathcal{P}(\mathcal{L}^+(A, \mathrm{i}_A)), L_{\mathcal{P}})$ endowed with Markov measures coming from the presented geometrical construction. The results will also be apply to more general Markov measures, the ones of *full support*, where non-empty cylinders in the shift space $\mathcal{P}(\mathcal{L}^+(A, \mathrm{i}_A))$ have positive measure.

In this context, there has been gathered sufficient information in Chapter 2 on connection properties for finite connected edge-indexed graphs without dead ends. Yet this last condition can be dropped. A bi-infinite geodesic $g \in \mathcal{G}(A, \mathrm{i}_A)$, does not have dead ends in the edge sequence. So there is a subgraph $A' < A$ having the same shift space $\mathcal{G}(A', \mathrm{i}_{A'}) = \mathcal{G}(A, \mathrm{i}_A)$ (we choose $\mathrm{i}_{A'} = \mathrm{i}_A|_{\mathrm{E}A'}$) and such that $(A', \mathrm{i}_{A'})$ has no dead ends. In this sense we can consider the graph $(A', \mathrm{i}_{A'})$ instead of (A, i_A) without changing anything in the dynamical system.

Thus we assume that (A, i_A) is as in (2.4). The Markov matrix of the shift space is given by

$$M_{a,b} = \begin{cases} 1 & \text{if } a, b \text{ is a geodesic in } (A, \mathrm{i}_A) \\ 0 & \text{if } a, b \text{ is not a geodesic in } (A, \mathrm{i}_A). \end{cases}$$

In case the corresponding topological Markov chain $(\mathcal{P}(\mathcal{L}^+(A, \mathrm{i}_A)), L_{\mathcal{P}})$ shall be equipped with a measure coming from the geometrical constructions on the universal cover, written in sections 5.5 and 5.6, we have to assume, that (A, i_A) is uni-modular. Then by Corollary 2.13 and by Proposition 5.16 there is a unique strictly positive eigenvector $[F]$ with eigenvalue e^{α} for the matrix $[R]$ given for indices $\mathbf{a}, \mathbf{b} \in \mathrm{E}A$ by

$$[R]_{\mathbf{a},\mathbf{b}} = \begin{cases} \mathrm{i}_A(\mathbf{b}) & \text{if } \mathrm{o}(\mathbf{b}) = \mathrm{t}(\mathbf{a}) \text{ and } \mathbf{b} \neq \overline{\mathbf{a}} \\ \mathrm{i}_A(\mathbf{b}) - 1 & \text{if } \mathbf{b} = \overline{\mathbf{a}} \\ 0 & \text{otherwise.} \end{cases}$$

This matrix was written in equation (5.28). There are exceptions only when $(A, \mathrm{i}_A) = (\mathrm{circ}_N, 1)$, still in these cases strictly positive eigenvectors exist (confer Example 5.18). If a strictly positive eigenvector $[F]$ is fixed, the construction of a measure can be carried out, writing for $a, b \in \mathrm{E}A$ the stochastic matrix

$$P_{a,b} = e^{-\alpha} \frac{F(b)}{F(a)} [R]_{a,b}$$

as justified in Proposition 5.19, as well as the stationary probability vector

$$p_a = \frac{F(\overline{a}) F(a)}{m(a)}$$

as introduced in Proposition 5.20. The stochastic matrix P and its stationary probability vector p satisfy naturally

$$\begin{gathered} M_{a,b} = 1 \quad \Longleftrightarrow \quad P_{a,b} > 0 \\ p_a > 0 \end{gathered} \tag{5.46}$$

for all edges $a, b \in \mathrm{E}A$. This condition is equivalent with full support of the Markov measure $\mu_{P,p}$.

In what follows, a graph (A, i_A) is supposed to cope with (2.4). A stochastic matrix P and a stationary probability vector p for P shall be chosen that satisfy (5.46). This allows to write a Markov measure $\mu_{P,p}$ on the shift space $\mathcal{P}(\mathcal{L}^+(A, \mathrm{i}_A))$ invariant under the shift operator L_{Pos}. This system shall be abbreviated as $(L_{Pos}, \mu_{P,p})$.

A consequence of $(M_{a,b} = 1) \Leftrightarrow (P_{a,b} > 0)$ for all $a, b \in \mathrm{E}A$ is that for any triple $a, b \in \mathrm{E}A$, $n \in \mathbb{N}$ one obtains

$$(M^n)_{a,b} > 0 \quad \Longleftrightarrow \quad (P^n)_{a,b} > 0. \tag{5.47}$$

The implication from the right-hand side to the left-hand side in shown in equation (5.30). As for the converse direction, using the positive number $m_p := \min\{P_{a,b} : P_{a,b} > 0, \quad a, b \in \mathrm{E}A\}$, one has $0 \leq m_p M_{a,b} \leq P_{a,b}$ for all $a, b \in \mathrm{E}A$, thus $0 < (M^n)_{a,b}$ implies $0 < (m_p M)^n{}_{a,b} \leq (P^n)_{a,b}$ by the same equation.

A square matrix $B \in \mathbb{R}^{|\mathrm{E}A| \times |\mathrm{E}A|}$ with non-negative entries is called *irreducible* if and only if for all indices $a, b \in \mathrm{E}A$ there is $n \in \mathbb{N}$ such that $(B^n)_{a,b} > 0$. The matrix B is called *transitive* if and only if there is $n \in \mathbb{N}$ such that $(B^n)_{a,b} > 0$ for all $a, b \in \mathrm{E}A$.

The classification of edge-indexed graphs is now prepared. An invariant Borel probability measure μ for a map L_{Pos} is called *ergodic*, if any invariant Borel measurable set S satisfies $\mu(S) = 0$ or $\mu(S) = 1$. We had defined in Section 2.4 the set of graphs

$$\begin{aligned} \mathcal{BG} &= \{(\mathrm{circ}_N, \mathrm{i}) : \mathrm{i}([j, j+1]) = 1, j \in \mathbb{Z}_N, N \in \mathbb{N}\} \\ &\quad \cup \quad \{(\mathrm{circ}_N, \mathrm{i}) : \mathrm{i}(\overline{[j, j+1]}) = 1, j \in \mathbb{Z}_N, N \in \mathbb{N}\}. \end{aligned}$$

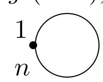
Theorem 5. *Given an edge-indexed graph (A, i_A) as in* (2.4)*, a stochastic matrix P and a stationary probability vector p for this matrix satisfying* (5.46)*, the dynamical system $(L_{Pos}, \mu_{P,p})$ is ergodic if and only if $(A, \mathrm{i}_A) \notin \mathcal{BG}$.*

If in addition (A, i_A) is unimodular, then $(L_{Pos}, \mu_{P,p})$ is ergodic if and only if $(A, \mathrm{i}_A) \neq (\mathrm{circ}_N, 1)$ for all $N \in \mathbb{N}$.

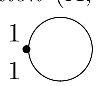
Proof. A series of equivalences gives the result. By Corollary 2.9, the graph (A, i_A) is not from $\mathcal{BG}$ if and only if it is a graph of $\mathcal{NG}$. By Theorem 1, the graph (A, i_A) is in $\mathcal{NG}$ if and only it is irreducible. This is the same as demanding for each two vertices a, b of $\mathcal{L}^+(A, \mathrm{i}_A)$ the existence of a positive path of positive length from a to b. By Lemma 1.9.4 in [3] the number of paths from a to b of length n are given by the number $(M^n)_{a,b}$. This shows that (A, i_A) is irreducible if and only if M is irreducible. This in turn is equivalent to P being irreducible through equation (5.47). Finally by Theorem 1.2 in [21] P is irreducible if and only if $(L_{Pos}, \mu_{P,p})$ is ergodic.

Under the condition of unimodularity, the first two equivalences in the proof can be replaced by Corollary 2.13. $\square$

An invariant probability measure μ for the map L_{Pos} is called *mixing* if for every pair of measurable sets B, C the property $\lim_{n\to\infty} \mu(B \cap f^{-n}C) = \mu(B)\mu(C)$ holds.

Theorem 6. *Given an edge-indexed graph* (A, i_A) *as in* (2.4)*, a stochastic matrix* P *and a stationary probability vector* p *for this matrix satisfying* (5.46)*, the dynamical system* $(L_{Pos}, \mu_{P,p})$ *is mixing if and only if* $(A, \mathrm{i}_A) \neq$ 

for all $n \in \mathbb{N}$ *and there are two closed geodesics of coprime lengths in* (A, i_A)*.*

If in addition (A, i_A) *is unimodular, then* $(L_{Pos}, \mu_{P,p})$ *is mixing if and only if* $(A, \mathrm{i}_A) \neq$ 

and there are two closed geodesics of coprime lengths in (A, i_A)*.*

Proof. The proof is structured as the proof of Theorem 5. By Theorem 2, a graph (A, i_A) as in (2.4) is transitive if and only if the condition above holds. We can argue as in Theorem 5 that transitivity of the graph implies transitivity of M. Conversely, if M is transitive, then for some $N \in \mathbb{N}$ holds $(M^N)_{a,b} > 0$ for all $a, b \in \mathrm{E}A$. We have to show, that also $(M^{N+1})_{a,b} > 0$ for all a, b. One can assume the opposite, $(M^{N+1})_{a,c} = \sum_b (M^N)_{a,b} M_{b,c} = 0$ for a pair $a, c \in \mathrm{E}A$. As there are no dead ends in (A, i_A), one has $M_{b',c} > 0$ for some $b' \in \mathrm{E}A$. Now the equation $0 = (M^N)_{a,b'} M_{b',c}$ implies $(M^N)_{a,b'} = 0$ and contradicts transitivity of M. The equivalence is complete. As a next step of the proof, equation (5.47) shows that transitivity of M is equivalent with transitivity of P. The demonstration is finished by Theorem 1.3 in [21] which states, that $(L_{Pos}, \mu_{P,p})$ is mixing is and only is P is transitive[2] .

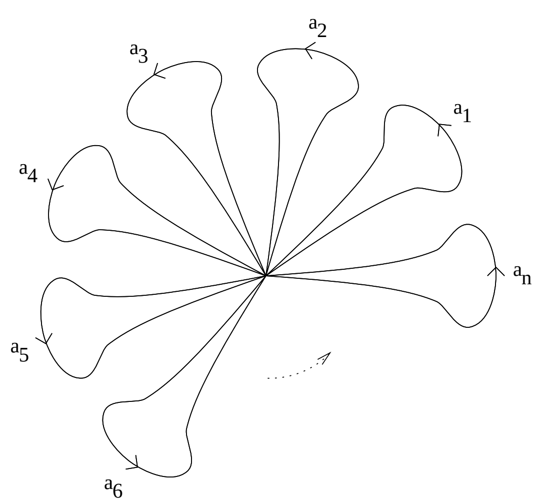
[2] The definition in [21] for a square matrix to be irreducible and aperiodic is equivalent with our definition of a matrix to be transitive, which is also used in the book [3] of A.Katok and B.Hasselblatt.

In the case of unimodularity, the first step of the proof can be replaced by Corollary 2.17. □

5.28 Example. In these examples we consider dynamical systems with shift spaces $\mathcal{P}(\mathcal{L}^+(A, \mathrm{i}_A))$ over a graph (A, i_A) and with shift operator $L_{\mathcal{P}}$. The dynamical system on a graph is supposed to carry a Markov measure $\mu_{P,p}$ in such a way that equation (5.46) is fulfilled.

- For the examples in Section 2.6 the questions for irreducibility and transitivity have been answered. A dynamical system on one of these graphs is ergodic if and only if the graph is irreducible (Theorem 5). It is mixing if and only if the graph is transitive (Theorem 6).
- Star graphs (confer Example 5.17) are trees. As shown in the last paragraph of Example 2.6, trees are irreducible and not transitive. Thus the dynamical system defined on a finite star graph is ergodic and not mixing.
- Considering the graph indicated in Figure 5.8 for $n \geq 2$, the dynamical

Figure 5.8: Diagram of a "fast star graph"



system exhibits a mixing property.

Appendices

Appendix A

Graphs

By definition, a graph A is called *bipartite*, if there are two subsets X_1, X_2 of the vertex set of A, such that $\mathrm{V}A$ is the disjoint union $\mathrm{V}A = X_1 \sqcup X_2$ and every edge $e \in \mathrm{E}A$ has a border vertex in each set X_1 and X_2.

There is a proof in [14], stating that a combinatorial graph A is bipartite, if and only if there is no closed path of odd length in A. For our purpose we need a slight generalization of this Theorem:

A.1 Corollary. *A graph A is bipartite, if and only if there is no closed path of odd length in A.*

Proof. We make use of a quotient graph, the graph A' derived from A by identifying all "multiple edges" to single ones. Before doing so, we have to exclude the case of loops included in A. If the edge e is a loop, then A is not bipartite because $\mathrm{o}(e) = \mathrm{t}(e)$ and also e is a closed path of length one in concordance with the statement.

If there are no loops, we define $G := \{g \in \mathrm{Aut}\,(A) : g(x) = x \text{ for all } x \in \mathrm{V}A\}$. Clearly G is a subgroup of $\mathrm{Aut}\,(A)$ as an intersection of vertex stabilizers. G also acts without inversion on A. If there is an inversion given by $h(a) = \overline{a}$, then $\mathrm{t}(a) = \mathrm{t}(\overline{\overline{a}}) = \mathrm{t}(\overline{h(a)}) = h(\mathrm{t}(\overline{a})) = \mathrm{t}(\overline{a}) = \mathrm{o}(a)$ uncovers the edge a as a loop. We can form the quotient graph $A' = G\backslash A$ with projection π. Vertices of A and A' will be identified via π.

The graph A' is a combinatorial graph. A loop πe in A' would give $\mathrm{t}(e) = \pi(\mathrm{t}(e)) = \mathrm{t}(\pi e) = \mathrm{o}(\pi e) = \pi(\mathrm{o}(e)) = \mathrm{o}(e)$. This is a contradiction because A

has no loops. If there was a circuit of length two in A', say with positive edges given by $\pi a, \pi b$, then $\mathrm{o}(a) = \pi(\mathrm{o}(a)) = \mathrm{o}(\pi a) = \mathrm{t}(\pi b) = \pi(\mathrm{t}(b)) = \mathrm{t}(b)$ as well as $\mathrm{o}(b) = \mathrm{t}(a)$. Then there is $g \in G$ with $b = g\overline{a}$, hence $\pi b = \overline{\pi a}$. But this is impossible, as a circuit of length two is an isomorphic image of circ_2.

The existence of a closed path of some length n in A implies the existence of a closed path of length n in A', since π is a morphism. The converse is also true. By means of Lemma 1.29 a path q in A' gives raise to a path p ($q = \pi \circ p$) in A. If q is closed, then p is closed by injectivity of π on vertices.

The proof is completed with the equations

$$\begin{aligned} \mathrm{o}(e) &= \pi(\mathrm{o}(e)) = \mathrm{o}(\pi(e)) \\ \mathrm{t}(e) &= \pi(\mathrm{t}(e)) = \mathrm{t}(\pi(e)) \end{aligned}$$

hat hold for all edges e of $\mathrm{E}A$. A bipartition for A' is a bipartition for A, a bipartition for A is one for A'. □

Appendix B

Groups

Let G be a *group* (see for example [16] for an introduction). If S is a subgroup of G we write $S < G$, which can mean equality as well. A *right coset* of $S < G$ in G is a subset of the form $St \subset G$ for $t \in G$. We write in short St for $\{st : s \in S\}$. The set of right cosets of S in G is denoted by

$$S \backslash G.$$

A *left coset* of S in G is a subset of the form $tS \subset G$ for some $t \in G$ ($tS = \{ts : s \in S\}$). The set of left cosets of S in G is denoted by

$$G/S.$$

We say that t is a *representative* of tS (and also of St).

If S is a subgroup of G, the *index* of S in G, denoted $[G : S]$, is the number of right cosets of S in G. By Theorem 2.7 in [16], the number of right cosets is equal to the number of left cosets. Given $x \in G$, a map $g \mapsto xgx^{-1}$ is called a *conjugation*. Every conjugation $G \to G$ is a *group-automorphism* of G, in particular a bijective map $G \to G$ [17].

B.1 Lemma. *Suppose there are two groups $H < G$ then*

$$[G : H] = [sGs^{-1} : H] \text{ for all } s \in G.$$

Proof. We work with left cosets. We can write $[sGs^{-1} : H] = |\{gH : g \in sGs^{-1}\}| = |\{gH : g \in G\}| = [G : H]$, the second equality because conjugation by $g \in G$ defines a bijection on G. □

B.2 Lemma. *Suppose there are three groups $H < G < L$. Then*

$$[G : H] = [sGs^{-1} : sHs^{-1}] \text{ for all } s \in L.$$

Proof. We work with left cosets. Since a conjugation by s is a bijection on L, it defines a bijection between subsets of L. Therefore $|x\{hH : h \in G\}x^{-1}| = |\{hH : h \in G\}|$ and hence

$$\begin{aligned} [xGx^{-1} : xHx^{-1}] &= |\{gxHx^{-1} : g \in xGx^{-1}\}| \\ &\overset{g=xhx^{-1}}{=} |\{xhHx^{-1} : h \in G\}| \\ &= |x\{hH : h \in G\}x^{-1}| \\ &= [G : H]. \end{aligned}$$

□

G *acts* on a set M by a homomorphism $\alpha : G \longrightarrow S_M$ from G to the *symmetric group* on M (cf. [16]). Therefore we have bijections $\alpha_g : M \to M$, $m \mapsto (\alpha(g))(m)$ for all $g \in$ G and we write simply $g(m)$ or gm instead of $(\alpha(g))(m)$.

If G acts on a set M then for $m \in M$ the set $Gm := \{gm : g \in G\}$ is called the *orbit* of m. The *stabilizer* of m is the subgroup

$$G_m := \{g \in \Gamma : gm = m\}$$

of group elements fixing m.

Appendix C

Metric spaces

A *metric space* (X, d) consists of a set X and a function $\mathrm{d} : X \times X \to \mathbb{R}$, with the properties

(i) $0 \leq \mathrm{d}(x, y) < \infty$ for all x, y,

(ii) $\mathrm{d}(x, y) = 0$ if and only if $x = y$,

(iii) $\mathrm{d}(x, y) = \mathrm{d}(y, x)$ for all x and y,

(iv) $\mathrm{d}(x, z) \leq \mathrm{d}(x, y) + \mathrm{d}(y, z)$ for all x, y, z.

A sequence $\{x_i\}_{i \in \mathbb{N}_0}$ in X is said to *converge in* X if and only if there is $x \in X$ such that for all $\epsilon > 0$ there is $N \in \mathbb{N}_0$ such that for all $n \in \mathbb{N}_0$ $\mathrm{d}(x_{N+n}, x) < \epsilon$. $\{x_i\}_{i \in \mathbb{N}_0}$ is called a *Cauchy sequence* if and only if for every $\epsilon > 0$ there is $N \in \mathbb{N}_0$ such that $\mathrm{d}(x_{N+k}, x_{N+l}) < \epsilon$ holds for all $k, l \in \mathbb{N}_0$. A metric space is called *complete* if and only if every Cauchy sequence in X converges in X.

An *open ball* of *radius* $r \in \mathbb{R}$ *about* a point $x \in X$ is the set $B_r(x) := \{y \in X : \mathrm{d}(x, y) < r\}$. The set of all open balls defines a topology for X (cf. Appendix D).

Appendix D

Topological spaces

We use to call a set of subsets of some set a *family* of subsets. Each element of a family is called a *member*. A subset of a family is also called a *subfamily*. A *topological space* is a set X in which a family τ of subsets (called *open sets*) has been specified with the following properties: X is open, $\emptyset$ is open, the intersection of any two open sets is open, and the union of of every subfamily of τ is open. The family τ is then called a *topology*. The topological space is also written (X, τ). A subset $C \subset X$ is called *closed* if and only if $X \setminus C$ is open.

A set $N \subset X$ is called a *neighborhood* of an element $x \in X$ if and only if N contains an open set to which x belongs. (X, τ) is called a *Hausdorff space* if and only if distinct points of X have disjoint neighborhoods.

A family $\mathcal{F} \subset \tau$ of open set is called an *open cover* for some set $Y \subset X$ if and only if Y is included in the union of the members of $\mathcal{F}$. A *subcover* of $\mathcal{F}$ for Y is any subfamily of $\mathcal{F}$, which is a cover of Y. A subset $Y \subset X$ is called *compact* if and only if each open cover of Y has a finite subcover. (X, τ) is called *locally compact* if and only if each point of X has a compact neighborhood.

A *base* of (X, τ) is a family $\mathcal{F}$ of open sets, such that every open set is a union of members of $\mathcal{F}$.

D.1 Example (Bases). The family of open spheres of a metric space form a base of a topology. This topology is called the *metric topology* (cf. [18] Chapter 4, Metric and Pseudo-metric Spaces).

Theorem 7 ([18], Theorem 1,11). *A family $\mathcal{B}$ of sets is a base for some* *topology for the set* $X = \bigcup_{B \in \mathcal{B}} B$ *if and only if for every two members* U *and* V *of* $\mathcal{B}$ *and each point* x *in* $U \cap V$ *there is* W *in* $\mathcal{B}$ *such that* $x \in W$ *and* $W \subset U \cap V$.

In case there are more than one topology under consideration for a given set X, say (X, τ) and (X, σ), we prefix all above introduced concepts by the name of the meant topology. For example a set could be τ-open but may not be σ-open.

If $Y \subset X$ and if σ is the family of all intersections $Y \cap U$, with $U \in \tau$, then (Y, σ) if a topology for Y, called the *relative topology inherited* from (X, τ). By definition a set $A \subset Y$ is σ-open if and only if it is the intersection of Y with a τ-open set. The same holds for closed sets (cf. [18], Chapter1, Relativation; Separation).

D.2 Lemma. *If* $\{B_\alpha\}_{\alpha \in I}$ *is a base for* (X, τ), *then* $A_\alpha := Y \cap B_\alpha$ *defines the members of a base for the relative topology of* (Y, σ) *inherited from* (X, τ).

Proof. Given U σ-open there is a τ-open set V such that $U = Y \cap V$. Hence $V = \bigcup_{\alpha \in J} B_\alpha$ some index set J. Then

$$U = Y \cap V = Y \cap \left(\bigcup_{\alpha \in J} B_\alpha \right) = \bigcup_{\alpha \in J} (Y \cap B_\alpha) = \bigcup_{\alpha \in J} A_\alpha.$$

□

To prove compactness for some subset $Y \subset X$ it is sufficient to consider open covers by members of some base of (X, τ): If $\mathcal{B}$ is a base for the topology of a space X such that every open cover of Y by members of $\mathcal{B}$ has a finite subcover for Y, then Y is compact (cf. [18] page 139).

D.3 Lemma. *Suppose* $Y \subset X$ *and* (Y, σ) *has the relative topology inherited from* (X, τ). *If* $K \subset Y$ *is* τ*-compact, then* K *is* σ*-compact.*

Proof. A τ-base $\{B_\alpha\}$ gives raise to a σ-base $\{A_\alpha\}$ by Lemma D.2. We consider an open cover for $Y \cap K$ by members of the σ-base $\{A_{alpha}\}_{\alpha \in J}$ for some index set J. $\{B_{alpha}\}_{\alpha \in J}$ is a τ-cover for K, hence has a finite subcover, say $B(1), \ldots, B(n)$.

$$K = K \cap Y \subset (B(1) \cup \ldots, B(n)) \cap Y = A(1) \cup \ldots \cup A(n)$$

show σ-compactness of K. □

For completeness, two more terms need to be introduced. A map F from a topological space A to a topological space B is called *continuous*, if the preimage of any open set is an open set. A group with some topology is called a *topological group*, if the group operations $g \mapsto g^{-1}$ and $(g, h) \mapsto gh$ are continuous. The topology of the domain of the second map is taken to be the *product topology*. This topology will be described, where necessary, through a base (see also [18],[19]).

Appendix E

Measures

Most terms concerning measures are taken from Halmos [22].

A *ring* of sets is a non empty family $\mathbf{R}$ of sets such that if $E \in \mathbf{R}$ and $F \in \mathbf{R}$, then $E \cup R \in \mathbf{R}$ and $E \setminus F \in \mathbf{R}$. A *σ-ring* is a non empty class $\mathbf{S}$ of sets such that

a) if $E \in \mathbf{S}$ and $F \in \mathbf{S}$ then $E \setminus F \in \mathbf{S}$, and

b) if $E_i \in \mathbf{S}, i = 1, 2, \ldots$, then $\bigcup_{i=1}^{\infty} E_i \in \mathbf{S}$.

The σ-ring $\mathbf{S}(\mathbf{E})$ *generated* by any family $\mathbf{E}$ of sets is the smallest σ-ring containing $\mathbf{E}$.

A *set function* is a function whose domain of definition is a family of sets. A function whose range is $\mathbb{R} \cup \{\infty\}$ is called an *extended real valued function.* An extended real valued set function μ defined on a class $\mathbf{E}$ is called *countably additive*, if, for every sequence $\{E_n\}_{n\in\mathbb{N}}$ of disjoint sets in $\mathbf{E}$, whose union is also in $\mathbf{E}$, we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

A *measure* is an extended real valued, non negative, and countable additive set function μ, defined on a ring $\mathbf{R}$, such that $\mu(\emptyset) = 0$. The set $\bigcup_{E\in\mathbf{R}} E$ is called the *measure space* of μ, the sets of the the ring $\mathbf{R}$ are called *measurable sets.*

A map f from a measure space X to a measure space Y is called *measurable* if $f^{-1}(E)$ is measurable for all measurable sets of Y. In case $Y = X$, the measure μ is called an *invariant measure* for f if $\mu\big(f^{-1}(E)\big) = \mu(E)$ for all measurable sets E.

Suppose X is a locally compact Hausdorff space, $\mathbf{C}$ is the family of all compact subsets of X and $\mathbf{S}$ the σ-ring generated by $\mathbf{C}$. The members of $\mathbf{S}$ are called the *Borel sets* of X. A *Borel measure* is a measure μ defined on the family $\mathbf{S}$ of all Borel sets and such that

$$\mu(C) < \infty \tag{E.1}$$

for all compact sets $C \in \mathbf{C}$.

A *Haar measure* is a Borel measure μ in a locally compact topological group X, such that

$$\mu(U) > 0 \tag{E.2}$$

for every non-empty open Borel set U, and $\mu(xE) = \mu(E)$ for every Borel set E and all group elements $x \in X$.

Since we will not make use of the term *regular* a weaker version of a Theorem from [22] will suffice our purposes.

Theorem 8 (Halmos, Theorem 58,B). *In every locally compact topological group X there exists at least one Haar measure.*

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