On Stable Convex Sets

Colloquium

of the

Pure Mathematics Research Centre

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speaker

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Overview

A convex set is stable if the midpoint map $(x, y) \mapsto \frac{1}{2}(x + y)$ is open.

Section 1 and 3 follow the chronological development of the theory of stable compact convex sets during the 1970's as described by Papadopoulou, Jber. d. Dt. Math.-Verein (1982) 92. The theory includes work by Vesterstrøm, Lima, O'Brien, Clausing, and Papadopoulou, among others.

Section 2 reports on a theory of generalized compactness (μ-compactness) developed by Holevo, Shirokov, and Protasov in the first decade of the 21st century. Density matrices form a stable μ-compact convex set. Applications to the continuity of entanglement monotones and von Neumann entropy are mentioned.

Sections 4 and 5 describe problems in finite dimensions related to stability of the set of density matrices: Continuity of inference, ground state problems, geometry of reduced density matrices, and continuity of correlation quantities.

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The CE-property (“continuous envelope”)

Definition 1. $K, Y, \mathcal{A}$ are subsets of a locally convex Hausdorff space; $\mathcal{A}$ is closed and bounded, $C(\mathcal{A})$ is the set of bounded continuous real functions on $\mathcal{A}$, and $M_1^+(\mathcal{A})$ the space of regular Borel probability measures on $\mathcal{A}$ (weak topology); if $\mathcal{A}$ is convex, then $A(\mathcal{A})$ is the set of continuous affine real functions on $\mathcal{A}$; K is a compact convex set.

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if $\mathcal{A}$ is convex, then the lower envelope of $f \in C(\mathcal{A})$ is

$$\check{f} : \mathcal{A} \to \mathbb{R}, \qquad \check{f}(x) = \sup\{g(x) : g \leqslant f, g \in A(\mathcal{A})\},$$

the barycenter of $\mu \in M_1^+(K)$ is $b(\mu) = \int_K x \, \mathrm{d}\mu(x)$

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Theorem 1. [Vesterstrøm, J. London Math. Soc. **2** (1973) 289]
$b : M_1^+(K) \to K$ is open if and only if $f \in C(K) \Rightarrow \check{f} \in C(K)$.

On the proof of Theorem 1

Reminder. [Alfsen, Compact Convex Sets and Boundary Integrals, Berlin: Springer (1971)]

$$\check{f}(x) = \min\{f(\mu) : x = b(\mu), \mu \in M_1^+(K)\}, \qquad f \in C(K)$$

$M_1^+(K)$ is w^*-compact, $b : M_1^+(K) \to K$ is a continuous, affine, and surjective map, $C(K) \cong A(M_1^+(K))$

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Theorem 2. [Vesterstrøm, ibid] TFAE
a) ϕ is open
b) $\check{f}^\phi \in C(K)$ for all $f \in A(Y)$ $\qquad$ ($\check{f}^b = \check{f}$ proves Thm. 1)

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Remark (continuity of inference maps)

Definition 2. Let $\phi : Y \to K$ as before. Assume $f \in C(Y)$ has for all $x \in K$ a unique minimum on $\phi^{-1}(x)$ and define

$$\Psi : K \to Y, \quad \Psi(x) = \text{argmin}\{f(y) : y \in \phi^{-1}(x)\}.$$

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note: the inference map Ψ choses a point in each fiber of ϕ which is optimal in the sense of minimizing f, a ranking function; the optimal value is $f(\Psi(x)) = \check{f}^{\phi}(x) = \min\{f(y) : y \in \phi^{-1}(x)\}$

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Observation 1. [Continuity of inference] If $f \in C(Y)$ has a unique minimum in each fiber of ϕ, then

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Proof. use Thm. 2 c) and compactness of Y

Stability of compact convex sets

Def. 3. K is stable if $K \times K \to K$, $(x, y) \mapsto \frac{x+y}{2}$ is open.

note: relative topologies are used on K and $K \times K$

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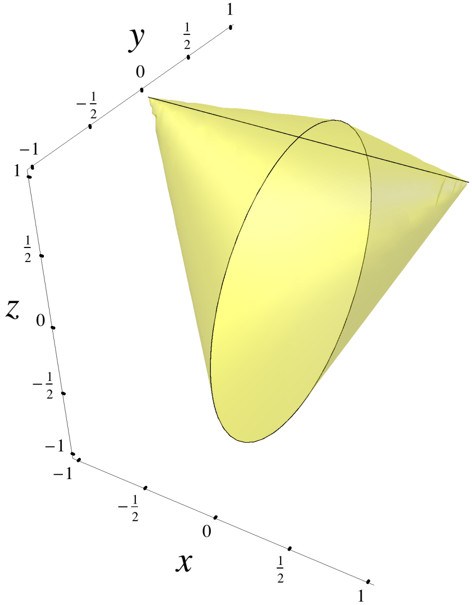
note: relative topologies are used on K and $K \times K$

Theorem 3. [O'Brien, Math. Ann. **223** (1976) 207] TFAE
a) the interior of every convex subset of K is convex
b) the convex hull of every open subset of K is open
c) K is stable
d) $\forall\lambda \in [0, 1]$: $K \times K \to K$, $(x, y) \mapsto (1 - \lambda)x + \lambda y$ is open
e) $K \times K \times [0, 1] \to K$, $(x, y, \lambda) \mapsto (1 - \lambda)x + \lambda y$ is open
f) the barycenter map $b : M_1^+(K) \to K$ is open

a)–e) are equivalent for general convex sets (Clausing and Papadopoulou, Math. Ann. **231** ('78) 193)

Standard example of a non-stable convex set



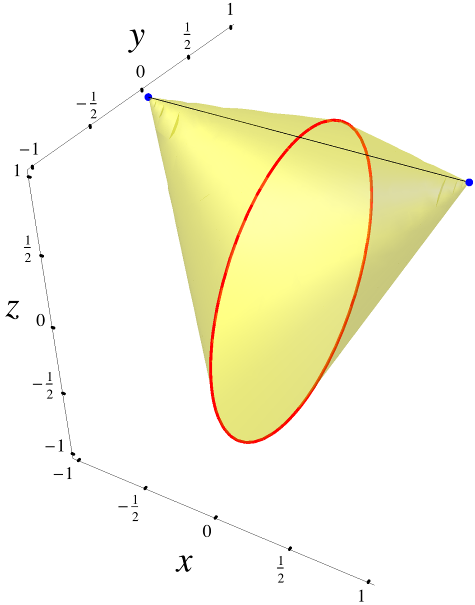
let K be the convex hull of the union of the circle

$$\{(0, y, z) : y^2 + z^2 = 1\}$$

and singletons

$$(\pm 1, 0, 1)$$

Example 1 a) Failure of the CE-property

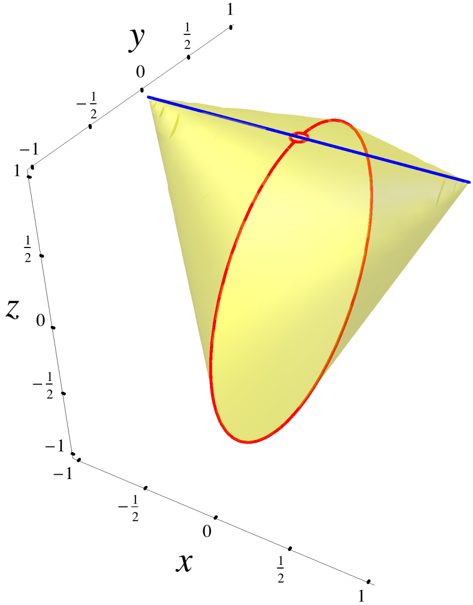


consider $f \in C(K)$

$f(x, y, z) = 1 - |x|$

$f(\textcolor{blue}{\bullet}) = 0$, $f(\textcolor{red}{\bullet}) = 1$

Example 1 a) Failure of the CE-property

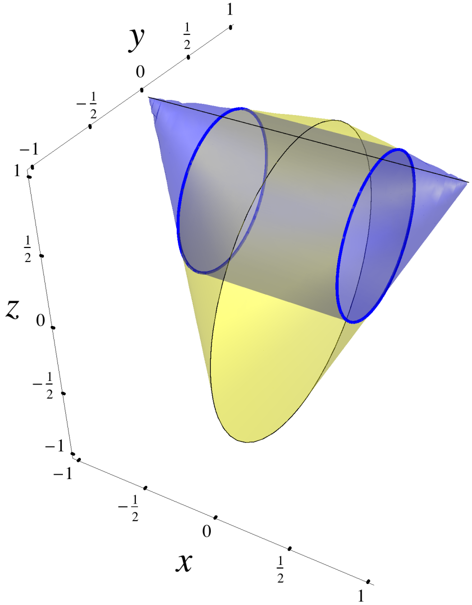


$\check{f}(a) = f(a)$ for all extreme points a of K

$\check{f}(\textcolor{blue}{\bullet}) = 0$, $\check{f}(\textcolor{red}{\bullet}) = 1$

$\Longrightarrow$ $\check{f}$ is discontinuous

Example 1 b) Non-convex interior of a convex set



consider the cylinder

$C = \{(x, y, z) :$
$y^2 + (z - \frac{1}{2})^2 \leqslant (\frac{1}{2})^2\}$

which extends in x-direction, and the convex set

$K \cap C$ (blue)

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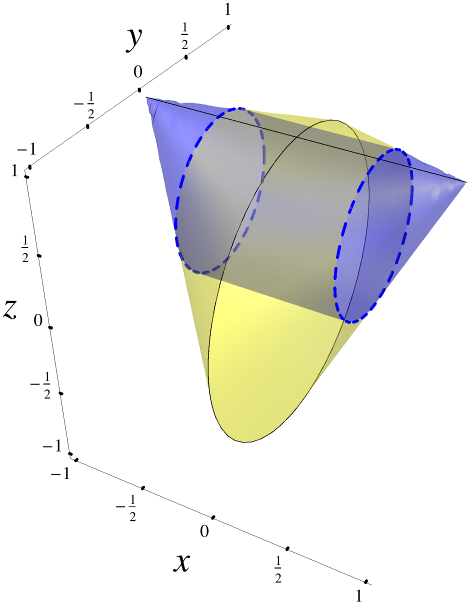


the boundary of $K \cap C$ is the surface

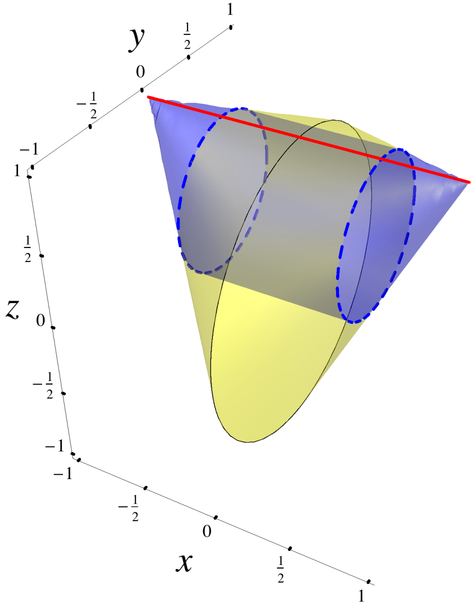
$\{(x, y, z) \in K :$

$|x| \leqslant \frac{1}{2}$,

$y^2 + (z - \frac{1}{2})^2 = (\frac{1}{2})^2\}$,

the interior of $K \cap C$ is depicted blue region

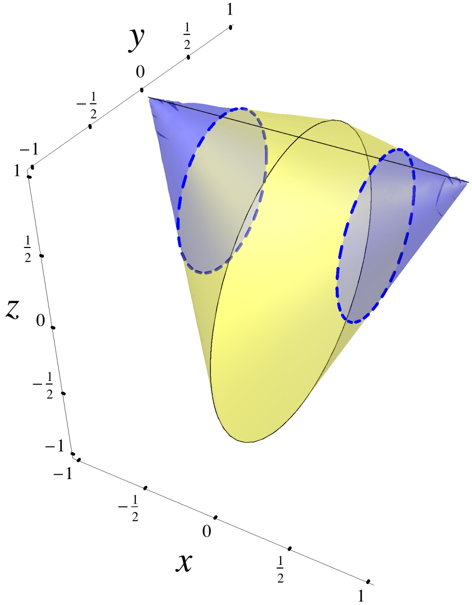
Example 1 b) Non-convex interior of a convex set



the red segment ends on both sides in the interior of $K \cap C$ (blue), but crosses the boundary of $K \cap C$

$\Longrightarrow$ the interior of $K \cap C$ is not convex

Example 1 c) Non-open convex hull of an open set



consider the open sets

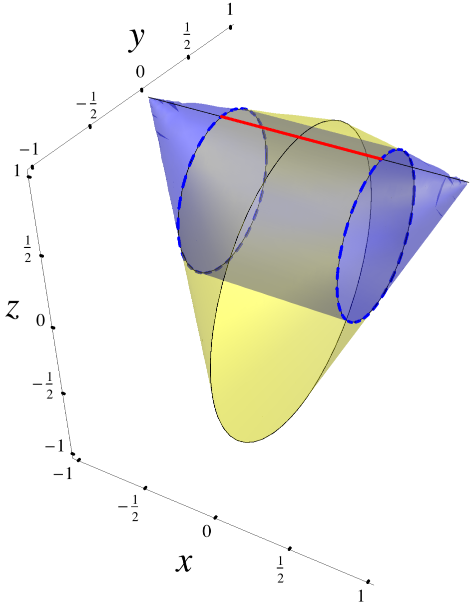
$O_\pm = \{(x, y, z) \in K :$

$\pm x > \frac{1}{2}\}$

and their union

$O = O_- \cup O_+$ (blue)

Example 1 c) Non-open convex hull of an open set



conv(O) is the union of the interior of $K \cap C$ (blue) and the red segment

$\implies$ conv(O) is not open

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Compact constraints on density matrices

how apply stability theory to density matrices?

let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{T}(\mathcal{H})$ the separable Banach space of trace-class operators on $\mathcal{H}$ with trace norm $\|A\|_1 = \operatorname{tr}\sqrt{A^*A}$

a density operator is a positive operator $\rho \in \mathfrak{T}(\mathcal{H})$ with $\operatorname{tr}(\rho) = 1$; the set $\mathfrak{S}(\mathcal{H})$ of density operators, the state space, is closed, bounded, and convex in $\mathfrak{T}(\mathcal{H})$

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an $\mathcal{H}$-operator is an unbounded positive operator H on $\mathcal{H}$ with discrete spectrum of finite multiplicity

Lemma 1. [Holevo & Shirokov, Theory Prob. Appl. **50** (2006) 86]
The set $\{\rho \in \mathfrak{S}(\mathcal{H}) : \operatorname{tr}(\rho H) \leqslant h\}$ is compact for every $\mathcal{H}$-operator H and $h < \infty$. For every compact subset $K \subset \mathfrak{S}(\mathcal{H})$ there exists an $\mathcal{H}$-operator H and $h < \infty$ such that $\operatorname{tr}(\rho H) \leqslant h$ for all $\rho \in K$.

μ-compact convex sets

$\mathfrak{S}(\mathcal{H})$ has a generalized compactness property

Definition 4. Let $\mathcal{A}$ be a closed bounded subset of a separable Banach space; for $\mu \in M_1^+(\mathcal{A})$ let

$$b(\mu) = \int_{\mathcal{A}} x \, \mathrm{d}\mu(x) \qquad \text{(integral in the sense of Bochner).}$$

$\mathcal{A}$ is μ-compact if the pre-image of every compact subset of $\overline{\mathrm{co}}(\mathcal{A})$ under $b : M_1^+(\mathcal{A}) \to \overline{\mathrm{co}}(\mathcal{A})$ is compact.

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Lemma 1 and Prokhorov's compactness theorem prove

Theorem 4. [Holevo and Shirokov, ibid] $\mathfrak{S}(\mathcal{H})$ is μ-compact.

Properties of μ-compact convex sets

let $\mathcal{A}$ be a μ-compact convex set, let $\mathrm{extr}(\mathcal{A})$ denote the set of extreme points of $\mathcal{A}$

Lemma 2. [Shirokov, Math. Notes **82** ('07) 395] For all $f \in C(\mathcal{A})$

$$\check{f}(x) = \min\{f(\mu) : x = b(\mu), \mu \in M_1^+(\mathcal{A})\}, \quad x \in \mathcal{A}.$$

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Lemma 3. [Protasov & Shirokov, Sbornik: Math. **200** ('09) 697]

$$\overline{\mathrm{co}}(\mathrm{extr}\,\mathcal{A}) = \mathcal{A} \quad \text{“Krein-Milman's theorem”}$$

$$b(M_1^+(\overline{\mathrm{extr}\,\mathcal{A}})) = \mathcal{A} \quad \text{“Choquet's theorem”}$$

Stability of density matrices

Theorem 5. [Shirokov, CMP **262** (2006) 137] $\mathfrak{S}(\mathcal{H})$ is stable.

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the Vesterstrøm-O'Brien theory generalizes to μ-compact convex sets

Theorem 6. [Protasov and Shirokov, ibid]
Let $\mathcal{A}$ be a convex μ-compact set. TFAE
a) $\mathcal{A}$ is stable
b) the barycenter map $b : M_1^+(\mathcal{A}) \to \mathcal{A}$ is open
c) the barycenter map $b : M_1^+(\overline{\operatorname{extr} \mathcal{A}}) \to \mathcal{A}$ is open
d) $f \in C(\mathcal{A}) \implies \check{f} \in C(\mathcal{A})$
Properties a)–d) imply $\overline{\operatorname{extr} \mathcal{A}} = \operatorname{extr} \mathcal{A}$.

Application to entanglement monotones

Let $f : \mathfrak{S}(\mathcal{H}) \to \mathbb{R}$ be concave. An entanglement monotone $E^f : \mathfrak{S}(\mathcal{K}) \to \mathbb{R}$ of a bi-partite system $\mathcal{K} = \mathcal{H} \otimes \mathcal{H}$ is defined by

$$E^f(\rho) = \inf\{\textstyle\sum_{i=1}^{\infty} \lambda_i f(\mathrm{tr}_2\, \rho_i) : \text{convex sum } \rho = \sum_{i=1}^{\infty} \lambda_i \rho_i,\ \rho_i \in \mathrm{extr}(\mathfrak{S}(\mathcal{K}))\}.$$

(Vidal, Plenio and Virmani, Osborne, etc.)

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Theorem 7. [Protasov and Shirokov, ibid]
Let $f \in C(\mathfrak{S}(\mathcal{H}))$ be concave. Then $E^f \in C(\mathfrak{S}(\mathcal{K}))$.

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Theorem 7. [Protasov and Shirokov, ibid]
Let $f \in C(\mathfrak{S}(\mathcal{H}))$ be concave. Then $E^f \in C(\mathfrak{S}(\mathcal{K}))$.

Proof. $E^f(\rho) \overset{a)}{=} \min\{f \circ \mathrm{tr}_2(\mu) : \rho = b(\mu), \mu \in M_1^+(\mathrm{extr}\,\mathfrak{S}(\mathcal{K}))\}$

$$\overset{b)}{=} \min\{f \circ \mathrm{tr}_2(\mu) : \rho = b(\mu), \mu \in M_1^+(\mathfrak{S}(\mathcal{K}))\} \overset{c)}{=} \overbrace{f \circ \mathrm{tr}_2}(\rho)$$

a) discrete measures are dense in
$\{\mu \in M_1^+(\mathrm{extr}\,\mathfrak{S}(\mathcal{K})) : \rho = b(\mu)\}$

b) $f \circ \mathrm{tr}_2$ is concave; c) Lemma 2

Application to von Neumann entropy

von Neumann entropy $S(\rho) = -\operatorname{tr} \rho \log(\rho)$, $\rho \in \mathfrak{S}(\mathcal{H})$

Remark. [Shirokov, Izvestiya: Math. **76** (2012) 840] Approx. technique for lower semi-continuous concave functions.

($\rightarrow$ necessary and sufficient continuity condition for S)

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let $\Phi : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{K})$ be a positive linear map;
the output entropy of Φ is $S_\Phi(\rho) = S(\Phi(\rho))$

Theorem 8. [Shirokov, `arXiv:1704.01905`] TFAE

a) Φ preserves continuity of S, i.e. for any $\rho_i \overset{i\to\infty}{\to} \rho \in \mathfrak{S}(\mathcal{H})$

$$S(\rho_i) \overset{i\to\infty}{\to} S(\rho) < \infty \implies S_\Phi(\rho_i) \overset{i\to\infty}{\to} S_\Phi(\rho) < \infty$$

b) Φ preserves finiteness of S, i.e. for any $\rho \in \mathfrak{S}(\mathcal{H})$

$$S(\rho) < \infty \implies S_\Phi(\rho) < \infty$$

c) S_Φ is bounded on the set $\operatorname{extr}\mathfrak{S}(\mathcal{H})$ of pure states

Remark (uniform continuity bounds for S)

Theorem 9. [Fannes, CMP **31** (1973) 291] $d := \dim(\mathcal{H}) < \infty$,

$$\tfrac{1}{2}\|\rho - \sigma\|_1 \leqslant \epsilon \leqslant 1 \implies |S(\rho) - S(\sigma)| \leqslant \epsilon\, d + h(\epsilon)$$

with binary entropy $h(x) = -x\log(x) - (1-x)\log(1-x)$.

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energy constraints are helpful if $\dim(\mathcal{H}) = \infty$

Theorem 10. [Winter, CMP **347** (2016) 191] Let H be an $\mathcal{H}$-operator such that $Z(\beta) := \mathrm{tr}(e^{-\beta H}) < \infty$ for all $\beta > 0$. If $E \geqslant 0$ and $\rho, \sigma \in \mathfrak{S}(\mathcal{H})$ such that $\mathrm{tr}(\rho H), \mathrm{tr}(\sigma H) \leqslant E$, then

$$\tfrac{1}{2}\|\rho - \sigma\|_1 \leqslant \epsilon \leqslant 1 \implies |S(\rho) - S(\sigma)| \leqslant \epsilon\, S(\gamma_{E/\epsilon}) + h(\epsilon)$$

where $\gamma_f = e^{-\beta_f H}/Z(\beta_f)$ has expected energy $f = \mathrm{tr}(\gamma_f H)$.

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Stability in finite dimensions

Definition 5. From now on $K \subset \mathbb{R}^n$ is a compact convex subset. The face function (Klee) of K is the multi-valued map $F_K : K \to K$, $F_K(x) = \bigcup_{y,z \in K, x \in]y,z[} [y, z]$.

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F_K is lower semi-continuous at $x \in K$ if $\forall y \in F_K(x)$ and $\forall$ open $V \ni y$ $\exists$ an open $U \ni x$ such that $x' \in U \Rightarrow F_K(x') \cap V \neq \varnothing$

a function $f : K \to \mathbb{R}$ is l.s.c. at $x \in K$ if $\forall \epsilon > 0$ $\exists$ a neighborhood U of x such that $x' \in U \Rightarrow f(x') > f(x) - \epsilon$

Stability in finite dimensions

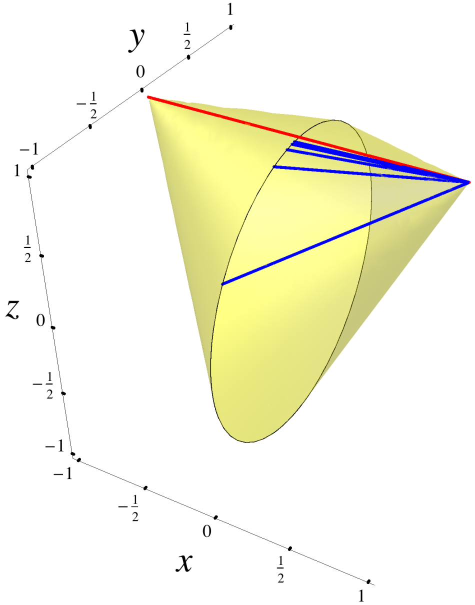
Definition 5. From now on $K \subset \mathbb{R}^n$ is a compact convex subset. The face function (Klee) of K is the multi-valued map $F_K : K \to K$, $F_K(x) = \bigcup_{y,z \in K, x \in]y,z[} [y, z]$.

F_K is lower semi-continuous at $x \in K$ if $\forall y \in F_K(x)$ and $\forall$ open $V \ni y$ $\exists$ an open $U \ni x$ such that $x' \in U \Rightarrow F_K(x') \cap V \neq \emptyset$

a function $f : K \to \mathbb{R}$ is l.s.c. at $x \in K$ if $\forall \epsilon > 0$ $\exists$ a neighborhood U of x such that $x' \in U \Rightarrow f(x') > f(x) - \epsilon$

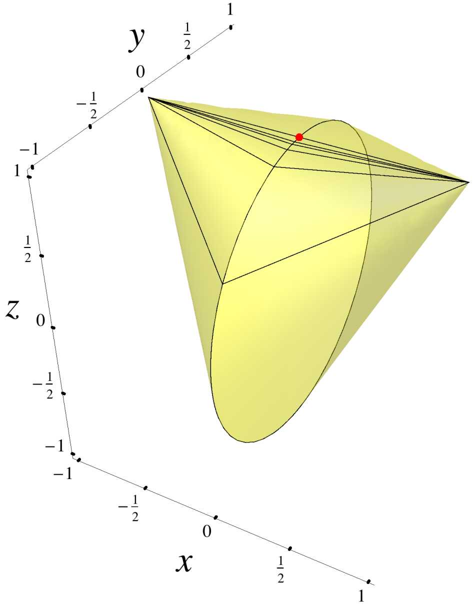
Theorem 11. [Papadopoulou, Math. Ann. **229** ('77) 193] TFAE
a) K is stable
b) F_K is lower semi-continuous
c) $\dim(F_K)$ is *l.s.c.*

Example 1 d) Failure of lower semi-continuity



the face function F_K fails to be lower semi-continuous on the red segment (except the endpoints)

Example 1 d) Failure of lower semi-continuity



the dimension function $\dim(F_K)$ fails to be l.s.c. at the red point

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Inference under linear constraints

Definition 2'. Consider the inference map $\Psi : K \to Y$,

$$\Psi(x) = \operatorname{argmin}\{f(y) : y \in \phi^{-1}(x)\},$$

defined by a surjective affine map $\phi : Y \to K$ and $f \in C(Y)$ which has a unique minimum in each fiber of ϕ.

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Quantum inference. $\mathcal{H} \cong \mathbb{C}^n$
Let $\langle a, b\rangle := \mathrm{tr}(a^* b)$ denote Hilbert-Schmidt inner product,
$M_n^{\mathrm{h}} := \{a \in M_n : a^* = a\}$, $U \subset M_n^{\mathrm{h}}$ a subspace,
and $\pi_U : M_n^{\mathrm{h}} \to M_n^{\mathrm{h}}$ the orthogonal projection onto U.
Define $Y = \mathfrak{S} = \mathfrak{S}(\mathbb{C}^n)$, $\phi = \pi_U|_{\mathfrak{S}}$, and $K = \phi(Y) = \pi_U(\mathfrak{S})$.

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Equivalently, replace U with $F_1, \ldots, F_k \in M_n^{\mathrm{h}}$ and π_U with the map $\mathbb{E} : M_n^{\mathrm{h}} \to \mathbb{R}^k$, $a \mapsto \langle a, F_i \rangle_{i=1}^k$.
$\mathbb{E}(\mathfrak{S})$ is the joint algebraic numerical range of $F_1, \ldots, F_k$.

Maximum-entropy inference

relative entropy $S(\rho, \sigma) = \operatorname{tr} \rho(\log(\rho) - \log(\sigma))$ of $\rho, \sigma \in \mathfrak{S}$,
$S(\rho, \sigma) = +\infty$ if $\rho(\mathcal{H}) \not\subset \sigma(\mathcal{H})$ (asymmetric distance)

Definition 6. Let an invertible state $\sigma \in \mathfrak{S}$ be fixed, let $\Psi_{U,\sigma} : \pi_U(\mathfrak{S}) \to \mathfrak{S}$ denote quantum inference with respect to the ranking function $f_\sigma(\rho) = S(\rho, \sigma)$.

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$\Psi_{U,\mathbb{1}/n}$ is maximum-entropy inference, since
$\log(n) - S(\rho, \frac{\mathbb{1}}{n}) = S(\rho) = -\operatorname{tr}\rho\log(\rho)$ is von Neumann entropy;
exponential family $\mathcal{F} = \mathcal{F}_{U,\sigma} := \{\frac{e^{\theta+u}}{\operatorname{tr} e^{\theta+u}} : u \in U\} \subset \mathrm{image}(\Psi_{U,\sigma})$
if $\theta := \log(\sigma)$, can we say more?

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if $\theta := \log(\sigma)$, can we say more?

- $\mathrm{d}_X : \mathfrak{S} \to [0,\infty], \mathrm{d}_X(\rho) = \inf_{\tau\in X} S(\rho,\tau)$, entropy distance from $X \subset \mathfrak{S}$
- $\widetilde{X} := \{\rho \in \mathfrak{S} : \mathrm{d}_X(\rho) = 0\}$, reverse information closure, Csiszár and Matúš, IEEE Trans. Inf. Theory **49** (2003) 1474

Entropic inference *via* reverse information topology

Theorem 12. [W, JCA **21** (2014) 339] For all $a \in \mathfrak{S} + U^{\perp}$ there is a unique $\pi_{\mathcal{F}}(a) \in (a + U^{\perp}) \cap \widetilde{\mathcal{F}}_{U,\sigma}$. For all $\rho \in \mathfrak{S}$, $\tau \in \widetilde{\mathcal{F}}_{U,\sigma}$

a) $S(\rho, \tau) = S(\rho, \pi_{\mathcal{F}}(\rho)) + S(\pi_{\mathcal{F}}(\rho), \tau)$ (Pythagorean thm.)

b) $\mathrm{d}_{\mathcal{F}}(\rho) = \mathrm{d}_{\widetilde{\mathcal{F}}}(\rho) = S(\rho, \pi_{\mathcal{F}}(\rho))$ (projection theorem)

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a) shows that the image of $\Psi_{U,\sigma}$ is $\widetilde{\mathcal{F}}_{U,\sigma}$;
hence, $\Psi_{U,\sigma}$ is continuous if and only if $\widetilde{\mathcal{F}}$ is norm closed
(notice that image and graph of $\Psi_{U,\sigma}$ are homeomorphic)

a) and b) show $\mathrm{d}_\mathcal{F}(\rho) = S(\pi_\mathcal{F}(\rho)) - S(\rho)$ for all $\rho \in \mathfrak{S}$

Discontinuity of maximum-entropy inference $\Psi_{U,\mathbb{1}/n}$

Example. Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, real *-algebra, $R = \mathrm{span}\{\sigma_1 \oplus 0, \mathrm{i}\,\sigma_2 \oplus 0, \sigma_3 \oplus 0, 0 \oplus 1\}$

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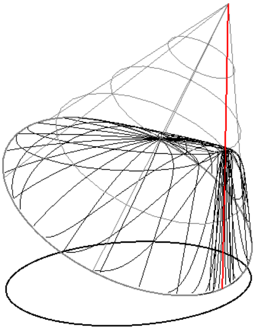
$U = \mathrm{span}\{\sigma_1 \oplus 1, \sigma_3 \oplus 0\}$, $F_1 = \sigma_1 \oplus 1$, $F_2 = \sigma_3 \oplus 0$,
circle of pure states $\rho_\alpha = \frac{1}{2}(\mathbb{1} + \cos(\alpha)\sigma_1 + \sin(\alpha)\sigma_3) \oplus 0$



the cone is the state space $\mathfrak{S}(R)$,
the ellipse below is $\pi_U(\mathfrak{S}(R))$,
the surface in $\mathfrak{S}(R)$ is $\mathrm{image}(\Psi)$,

the ρ_α's (base circle of $\mathfrak{S}(R)$) lie in $\mathrm{image}(\Psi)$ except for the bottom point ρ_0 of the red fiber of $\pi_U|_{\mathfrak{S}(R)}$

$\Rightarrow$ Ψ is discontinuous at $\pi_U(\rho_0)$

W. and Knauf, JMP **53** (2012) 102206

Continuity of Ψ *via* openness of the affine map ϕ

the continuity condition of Observation 1 has a local counterpart

Definition 7. The map $\phi : Y \to K$ is open at $y \in Y$ if $\phi(V)$ is a neighborhood of $\phi(y)$ for every neighborhood V of y.

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For each $x \in K$ the inference map Ψ is continuous at x if and only if ϕ is open at $\Psi(x)$.

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For each $x \in K$ the inference map Ψ is continuous at x if and only if ϕ is open at $\Psi(x)$.

K without reference to $\phi : Y \to K$ can witness openness of ϕ:

a) ϕ is open if K is a polytope, e.g. quantum inference where $F_1, \ldots, F_k$ are commutative

b) ϕ is open on all fibers of relative interior points of K

Disontinuity of Ψ for a stable Y

the relative interior $\mathrm{ri}(X)$ of X is the interior in the affine hull of X

Theorem 14. [Rodman, Szkoła, Spitkovsky, W, JMP **57** ('16)]
Let Y be stable and let $x \in K$ such that $\Psi(x) \in \mathrm{ri}(\phi^{-1}(x))$.
If a sequence $(x_i) \subset K$ converges to x and $\Psi(x_i) \to \Psi(x)$ for $i \to \infty$, then $\dim(F_K(x)) \leqslant \liminf_{i\to\infty} \dim(F_K(x_i))$.

Proof. use Papadopoulou's Thm. 11 and compare the face functions F_K and F_Y

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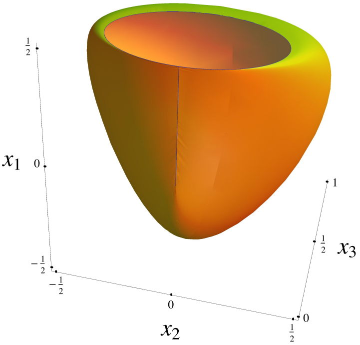
Example. Chien and Nakazato,
Lin. Alg. Appl. **432** (2010) 173,

reproduced from Szymański, W, and Życzkowski, `arXiv:1603.06569`

$F_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $F_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

the picture shows a surface whose convex hull is $\mathbb{E}(\mathfrak{S})$



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Ground state problems

Def. 8. The smallest eigenvalue $\lambda_0(a)$ of $a \in M_n^{\mathrm{h}}$ is the ground state energy of a, its spectral projection $p_0(a)$ the ground space projection. $\mathcal{P}(U) := \{p_0(u) : u \in U\} \cup \{0\}$.

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$$\lambda_0(u) = \min_{\rho \in \mathfrak{S}} \langle \rho, u \rangle = \min_{a \in \pi_U(\mathfrak{S})} \langle a, u \rangle, \; u \in U \qquad \text{(Toeplitz)}$$

Definition 9. An exposed face of a K is $\varnothing$ or a subset of the form $\operatorname{argmin}_{x \in K} \langle x, u \rangle$ for some vector u. The lattice of exposed faces of K is denoted $\mathcal{E}(K)$.

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- lattice isomorphism $\{p \in M_n : p = p^2 = p^*\} \cong \mathcal{E}(\mathfrak{S})$, $j(p) = \{\rho \in \mathfrak{S} : s(\rho) \preceq p\}$, support projection $s(\rho)$ (Kadison)
- lattice isomorphism $\mathcal{P}(U) \cong \mathcal{E}(\pi_U(\mathfrak{S}))$, isomorphism $\pi_U \circ j$

Ground state energy: Level crossings

let $F_1, F_2 \in M_n^{\mathrm{h}}$ and for all $\theta \in \mathbb{R}$ let $A(\theta) = \cos(\theta) F_1 + \sin(\theta) F_2$,

$$A(\theta) x_k(\theta) = \lambda_k(\theta) x_k(\theta) \qquad \text{(Rellich)}$$

where $\{x_k(\theta)\}_{k=1}^n$ is an ONB of $\mathbb{C}^n$ analytic in θ; consider curves

$$\begin{aligned} z_k(\theta) &= \langle x_k(\theta), (F_1 + \mathrm{i}\, F_2) x_k(\theta) \rangle = \mathbb{E}(|x_k(\theta)\rangle\langle x_k(\theta)|) \\ &= e^{\mathrm{i}\,\theta}(\lambda_k(\theta) + \mathrm{i}\, \lambda_k'(\theta)) \end{aligned}$$

in the numerical range $\mathbb{E}(\mathfrak{S})$

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Theorem 15. [W, Rep. Math. Phys. **77** (2016) 251,
Leake, Lins, and Spitkovsky, Lin. Mult. Algebra **62** (2014) 1335]
If z is an extreme point of $\mathbb{E}(\mathfrak{S})$ then there are k_0 and θ_0 such that $z = z_{k_0}(\theta_0)$. The map $\Psi_{F_1,F_2,\sigma}$ is continuous at z if and only if for all k such that $z = z_k(\theta_0)$ we have $\lambda_{k_0} = \lambda_k$.

context of quantum phase transitions: Chen, Ji, Li, Poon, Shen, Yu, Zeng, Zhou, New J. Phys. **17** (2015) 083019

Discontinuity of Ψ means $\mathcal{P}(U)$ is not closed

Example. $U = \text{span}\{\sigma_1 \oplus 1, \sigma_3 \oplus 0\}$,
$\mathcal{P}(U) \backslash \{0 \oplus 0, \mathbb{1} \oplus 1\} = \{\rho_\alpha : \alpha \in]0, 2\pi[\,\} \cup \{\rho_0 + 0 \oplus 1\}$

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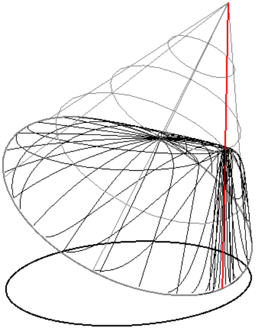

ρ_0 lies in the closure of $\mathcal{P}(U)$ but not in $\mathcal{P}(U)$

the maximum-entropy is discontinuous at $\pi_U(\rho_0)$

in the drawing, ρ_0 is the bottom point of the red fiber of $\pi_U|_{\mathfrak{S}(R)}$

W. and Knauf, JMP **53** (2012) 102206

Geometry of quantum marginals

a k-local Hamiltonian is a sum of hermitian matrices $a_1 \otimes \cdots \otimes a_N \in M_n^{\otimes N}$ each term at most k non-scalar factors a_i; denote the space of k-local Hamiltonians by U_k

Local Hamiltonian Problem. Given $u \in U_k$ and $(\xi - \eta) \propto 1/\text{poly}(N)$, determine whether the ground state energy $\lambda_0(u)$ is $> \xi$ or $< \eta$.

Zeng, Chen, Zhou, Wen, `arXiv:1508.02595`,
Cubitt and Montanaro, SIAM Journal on Computing **45** (2016) 268

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Geometric Problem. [Chen, Ji, Kribs, Wei, Zeng, JMP **53** (2012)]
The set of k-body marginals $\pi_{U_k}(\mathfrak{S}) \cong \{\mathrm{tr}_\nu(\rho)_{|\nu|=k} : \rho \in \mathfrak{S}\}$ encodes ground state energy $\lambda_0(u) = \min_{a \in \pi_{U_k}(\mathfrak{S})} \langle a, u \rangle$, $u \in U_k$;
goal: analyze exposed faces $\operatorname{argmin}_{a \in \pi_{U_k}(\mathfrak{S})} \langle a, u \rangle$ of $\pi_{U_k}(\mathfrak{S})$ and lattice of ground space projections $\mathcal{P}(U_k) \cong \mathcal{E}(U_k)$

Irreducible many-body correlation

exponential family $\mathcal{F}_k = \{\frac{e^u}{\operatorname{tr} e^u} : u \in U_k\}$ of k-local Hamiltonians

Definition 10. irreducible correlation $C_k(\rho) = \mathrm{d}_{\mathcal{F}_k}(\rho)$

C_k is the entropy distance from $\mathcal{F}_k$ and the difference of von Neumann entropies $C_k(\rho) = S(\pi_{\mathcal{F}_k}(\rho)) - S(\rho)$ (Thm. 12)

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C_k quantifies correlation/complexity which cannot be described by interactions between less than k particles; example $k = 1$:
mutual information $C_1(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$
multi-information $C_1(\rho_{ABC}) = S(\rho_A) + S(\rho_B) + S(\rho_C) - S(\rho_{ABC})$

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- statistics (Amari, IEEE Trans. Inf. Theory **47** (2001) 1701, Ay, Annals Prob. **30** (2002) 416)
- quantum information (Linden et al. ibid, Zhou, PRL **101** (2008) 180505, Niekamp et al. J. Physics A **46** (2013) 125301, W. et al. OSID **22** (2015) 1550006)

Example: 3-qubit 2-local Hamiltonians

Theorem 16. [Linden, Popescu, Wootters, PRL **89** (2002) 207901] If $|\psi\rangle \in (\mathbb{C}^2)^{\otimes 3}$ is not locally unitary equivalent to $\alpha|000\rangle + \beta|111\rangle$, then $\pi_{U_2}(\rho) = \pi_{U_2}(|\psi\rangle\langle\psi|) \Rightarrow \rho = |\psi\rangle\langle\psi|$.

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$\Rightarrow \pi_{U_2}|_{\mathfrak{S}}$ is open at pure states ρ which are not locally unitarily equivalent to $\alpha|000\rangle + \beta|111\rangle$
$\Rightarrow C_2(\rho) = 0$ and C_2 is continuous at ρ

C_2 is discontinuous at $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, where $C_2(|\mathrm{GHZ}\rangle) = 1$, Zhou, PRL **101** (2008) 180505]

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Theorem 14 and stability of $\mathfrak{S}$ explain the discontinuity of C_2 in terms of geometry:

$\pi_{U_2}(|GHZ\rangle\langle GHZ|)$ is the midpoint of a segment but is approximated by exposed points of $\pi_{U_2}(\mathfrak{S})$

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Conclusion

Stability of $\mathfrak{S}(\mathcal{H})$ provides analytic method to study continuity of information theoretic quantities (von Neumann entropy, entanglement monotones).

Stability of $\mathfrak{S}(\mathbb{C}^n)$ gives new insights into continuity of inference, ground state problems, geometry of reduced density matrices, and continuity of correlation quantities.

Thank you for the attention

Thanks to Maksim E. Shirokov (Moscow) and Andreas Winter (Barcelona) for discussions about infinite dimensions