A new signature of quantum phase transitions from the numerical range

talk at the conference

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From Physics to Information Sciences and Geometry

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speaker

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Overview

1. Ground Energy
2. Convex Geometry
3. Numerical Range
4. Results
5. Conclusion

Part I
Ground Energy

Quantum Phase Transitions

are characterized in terms of

1) long-range correlation in ground state

2) non-analytic ground energy

3) geometry of reduced density matrices
Zauner-Stauber et al. New J. Phys. 18 (2016), 113033
& Chen et al. Phys. Rev. A 93 (2016), 012309

4) strong variation / discontinuity of MaxEnt maps
Arrachea et al. Phys. Rev. A 45 (1992), 7104
& Chen et al. New J. Phys. 17 (2015), 083019

this talk clarifies in the finite-dimensional setting relationships between 2), 3), 4) and certain open mapping properties

Differentiability of Ground Energy

one-parameter Hamiltonian $H(g) = H_0 + g \cdot H_1, \quad g \in \mathbb{R}$

angular representation $A(\theta) = \cos(\theta) H_0 + \sin(\theta) H_1$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

ground energy $\lambda(X) =$ minimal eigenvalue of X

Observation

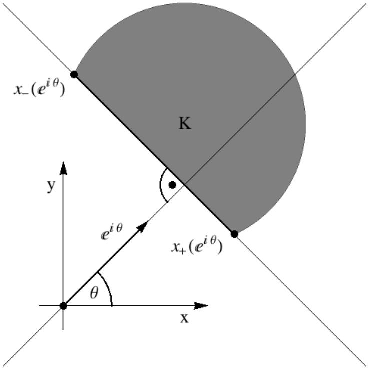
$\lambda \circ H$ is C^k/analytic at $\tan(\theta) \iff \lambda \circ A$ is C^k/analytic at θ

focus on ground energy $\lambda(\theta) = \lambda \circ A(\theta)$

advantage: reduced density matrices and MaxEnt maps are easier described in angular coordinates

Part II
Convex Geometry

Convex Geometry



$K \subset \mathbb{C} \cong \mathbb{R}^2$ compact convex

support function
$\tilde{h}_K : \mathbb{C} \to \mathbb{R}$, $\tilde{h}_K(u) = \min_{z \in K} \langle z, u \rangle$

exposed face $F(u) = \text{argmin}_{z \in K} \langle z, u \rangle$
$\tilde{h}_{F(u)}(v) = \tilde{h}'_K(u; v) =$ directional derivative $= \lim_{t \searrow 0} \frac{1}{t}(\tilde{h}_K(u + tv) - \tilde{h}_K(u))$

end-points of exposed face, $u \in S^1$
$x_\pm(u) = u\tilde{h}_K(u) \pm u^\perp \tilde{h}'_K(u; \pm u^\perp) \in \partial K$

using $h_K(\theta) = \tilde{h}_K(e^{i\theta})$,
$x_\pm(e^{i\theta}) = e^{i\theta}(h_K(\theta) \pm i\, h'_K(\theta; \pm 1)) \in \partial K$

$x_\pm$ restricted to $u \in S^1$ with $x_+(u) = x_-(u)$ is the reverse Gauss map $x_\pm(u) = \nabla \tilde{h}(u)$ which parametrizes ∂K as an envelope

Part III
Numerical Range

Numerical Range

density matrices $D_n = \{\rho \in M_n : \rho \geq 0, \mathrm{tr}(\rho) = 1\}$

numerical range $W = \{\langle\psi|H_0 + \mathrm{i}\,H_1|\psi\rangle : \langle\psi|\psi\rangle = 1\}$

$$= \{\mathrm{tr}\,\rho(H_0 + \mathrm{i}\,H_1) : \rho \in D_n\}$$

W is the set of expected values of H_0 and H_1 (reduced density matrices)

Theorem 1 (Toeplitz) $h_W(\theta) = \lambda \circ A(\theta) = \lambda(\theta)$

Math. Z. 2 (1918), 187

von Neumann entropy $S(\rho) = -\,\mathrm{tr}\,\rho\log(\rho)$, $\rho \in D_n$

maximum-entropy inference map (MaxEnt map)

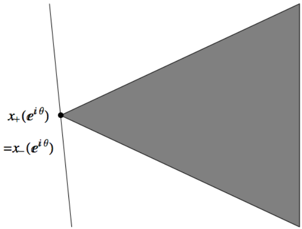
$$\rho^* : W \to D_n, \quad z \mapsto \mathrm{argmax}\{S(\rho) : \mathrm{tr}\,\rho(H_0 + \mathrm{i}\,H_1), \rho \in D_n\}$$

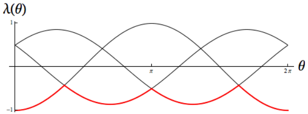
Numerical Range — Diagonal Matrices

$$H_0 = \mathrm{diag}(E_0^1, \ldots, E_0^n) \text{ and } H_1 = \mathrm{diag}(E_1^1, \ldots, E_1^n)$$

$$W = \mathrm{conv}\{E_0^1 + \mathrm{i}\, E_1^1, \ldots, E_0^n + \mathrm{i}\, E_1^n\}$$

$$A(\theta) = \mathrm{diag}(E_0^1 \cos(\theta) + E_1^1 \sin(\theta), \ldots, E_0^n \cos(\theta) + E_1^n \sin(\theta))$$





- W is a polytope
- λ is piecewise harmonic
- x_+ and x_- are piecewise constant
- flat boundary portions of W $\cong$ non-differentiable points of λ
- ρ^* is continuous

Numerical Range — Non-Commutative

we assume $\dim(W) = 2 \quad \Leftarrow \quad [H_0, H_1] \neq 0$

analytic curves $\lambda_1(\theta), \ldots, \lambda_n(\theta)$ and ONB's $|\psi_k(\theta)\rangle_{k=1}^{n}$ such that

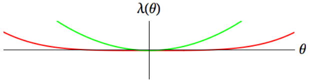
$$A(\theta) = \sum_{k=1}^{n} \lambda_k(\theta) \, |\psi_k(\theta)\rangle \, \langle\psi_k(\theta)|$$

Rellich, IMM-NYU 2, New York: New York University, 1954

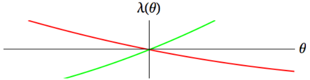
- λ is piecewise analytic

- the maximal order of differentiability of λ is even at non-analytic points

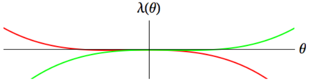
analytic



max. order 0



max. order 2



Numerical Range — Continuity of Inference

ρ^* is analytic on the interior of W, $W^\circ \ni z \mapsto e^{\mu_1 H_1 + \mu_2 H_2} / \mathrm{tr}('')$,
if $z = x_+(e^{\mathrm{i}\theta})$ then

$$\rho^*(z) = \text{ maximally mixed state on}$$
$$\mathrm{span}\{\, |\psi_k(\theta)\rangle : \lambda_k(\theta) = \lambda(\theta),\ \lambda'_k(\theta) = \lambda'(\theta; +1)\,\}$$

- the maps $x_+, x_- : S^1 \to \partial W$ cover all extreme points of W
- $\rho^*|_{\partial W}$ may be discontinuous at extreme points of W because of C^2 smooth eigenvalue crossings with the ground energy λ
- $\rho^*|_{F(u)}$ is continuous on flat boundary portions $F(u) \subset \partial W$ of W
- for $z \in \partial W$: $\rho^*|_{\partial W}$ is continuous at $z \iff \rho^*$ is continuous at z
- discontinuities of ρ^* **are irremovable** because $\rho^*(W) \subset \overline{\rho^*(W^\circ)}$, Wichmann JMP 4 (1963), 884
- discontinuities of $\rho^*|_{\partial W}$ **may be removable**

Numerical Range — Open Mappings

definitions

a map $\alpha : X \to Y$ between topological spaces is open at $x \in X$ if α maps neighborhoods of x to neighborhoods of $\alpha(x)$

numerical range map $f : \{|\psi\rangle : \langle\psi|\psi\rangle = 1\} \to W$, $|\psi\rangle \mapsto \langle\psi|H_0 + \mathrm{i}\, H_1|\psi\rangle$
expected value map $\mathbb{E} : D_n \to W$, $\rho \mapsto \mathrm{tr}\, \rho(H_0 + \mathrm{i}\, H_1)$

the inverse numerical range map f^{-1} is strongly (resp. weakly) continuous at $z \in W$ if **for all** (resp. **for at least one**) $|\psi\rangle \in f^{-1}(z)$ the map f is open at $|\psi\rangle$

Noticeable: Openness of linear maps on state spaces of C^*-algebras are studied since the 70's (Lima, Vesterstrøm, O'Brian), with applications to quantum information theory: Shirokov, Izvestiya: Math. 76 (2012), 840

Part IV
Results

Smoothness of λ, geometry of W, continuity of ρ^*

Let $z = x_+(e^{\mathrm{i}\,\theta}) = x_-(e^{\mathrm{i}\,\theta})$ be not a corner point: the statements in each column are equivalent

<table>
<tr><td colspan="2">λ is analytic locally at θ</td><td>λ is C^{2k} but not C^{2k+1} locally at θ, $k \geq 1$</td></tr>
<tr><td>$\lambda(\theta) = \lambda_k(\theta) = \lambda_l(\theta)$
$\lambda'(\theta) = \lambda'_k(\theta) = \lambda'_l(\theta)$
$\Rightarrow \lambda_k = \lambda_l$</td><td>$\exists k : \lambda = \lambda_k$ locally at θ</td><td>$\nexists\, k : \lambda = \lambda_k$ locally at θ</td></tr>
<tr><td colspan="2">∂W is an analytic manifold locally at z</td><td>∂W is a C^{2k} but not a C^{2k+1} manifold locally at z, $k \geq 1$</td></tr>
<tr><td>ρ^* is continuous at z</td><td>$\rho^*|_{\partial W}$ has a removable discontinuity at z</td><td>$\rho^*|_{\partial W}$ has an irremovable discontinuity at z</td></tr>
</table>

Notice: $S^1 \to \partial W$, $e^{\mathrm{i}\,\theta} \mapsto x_\pm(\theta) = e^{\mathrm{i}\,\theta}(\lambda(\theta) + \mathrm{i}\,\lambda'(\theta))$ is only C^{2k-1} if λ is C^{2k} !

Open Mapping Conditions

Let $z \in W$ be arbitrary: the statements in each column are equivalent

ρ^* is continuous at z	$\rho^*\|_{\partial W}$ has a removable discontinuity at z	$\rho^*\|_{\partial W}$ has an irremovable discontinuity at z
f^{-1} is strongly continuous at z	f^{-1} is weakly but not strongly continuous at z	f^{-1} is not weakly continuous at z
$\mathbb{E}$ is open at $\rho^*(z)$	$\mathbb{E}$ is not open at $\rho^*(z)$	

Part V
Conclusion

Summary:
Geometry and inference approach to the smoothness of the ground energy of a one-parameter Hamiltonian

References:

Weis and Knauf, *Entropy distance: New quantum phenomena*, JMP 53 (2012), 102206

Leake et al., *Inverse continuity on the boundary of the numerical range*, Linear and Multilinear Algebra 62 (2014), 1335

Rodman et al., *Continuity of the maximum-entropy inference: Convex geometry and numerical ranges approach*, JMP 57 (2016), 015204

Spitkovsky and Weis, *A new signature of quantum phase transitions from the numerical range*, `arXiv:1703.00201 [math-ph]`

Thank you