Classification of joint numerical ranges of three hermitian matrices of size three

talk at

The 14th Workshop on Numerical Ranges and Numerical Radii

Max-Planck-Institute MPQ, München, Germany

June 15th, 2018

speaker

Stephan Weis

Université libre de Bruxelles, Belgium

joint work with

Konrad Szymański and Karol Życzkowski

Jagiellonian University, Kraków, Poland

Overview

1. Introduction
2. Problems with 3D Joint Numerical Ranges of 3-by-3 Matrices
3. Solution: Graph Embedding (definition of classes)
4. Finding Examples (all classes are populated)
5. Conclusion

Introduction

Joint Numerical Ranges

let $F_1, \ldots, F_k \in M_d$ denote hermitian d-by-d matrices, the state space (mixed states) of a *-subalgebra $\mathcal{A} \subset M_d$ is

$$\mathcal{M}(\mathcal{A}) = \{\rho \in \mathcal{A} \mid \rho \geq 0, \mathrm{tr}(\rho) = 1\},$$

the joint algebraic numerical range of $F = (F_1, \ldots, F_k)$ is

$$L_F = \{(\mathrm{tr}(\rho F_1), \ldots, \mathrm{tr}(\rho F_k)) : \rho \in \mathcal{M}(M_d)\} \subset \mathbb{R}^k,$$

the joint numerical range (JNR) of F is

$$W_F = \{(\langle\psi|F_1\psi\rangle, \ldots, \langle\psi|F_k\psi\rangle) : |\psi\rangle \in \mathbb{C}^d, \langle\psi|\psi\rangle = 1\}$$

with $\langle\varphi|\psi\rangle = \overline{\varphi_1}\psi_1 + \cdots + \overline{\varphi_d}\psi_d$

Lemma

$$\mathrm{conv}(W_F) = L_F.$$

Convexity of Numerical Ranges

Theorem (Toeplitz and Hausdorff)

W_{F_1,F_2} is convex. Hence $W_{F_1,F_2} = L_{F_1,F_2}$.

Math. Z. 2 (1918), 187 and Math. Z. 3 (1919), 314

Theorem (Au-Yeung and Poon)

If $d \geq 3$ then W_{F_1,F_2,F_3} is convex. Hence $W_F = L_F$.

Southeast Asian Bull. Math. 3 (1979), 85

there is no easy rule to decide convexity of $W_{F_1,\ldots,F_k}$ if $k \geq 4$

Li and Poon, SIAM J. Matrix Anal. Appl. 21 (2000), 668

Boundary Generating Curve ($k = 2$)

consider the hypersurface

$$V_{F_1,F_2} = \{(u_0 : u_1 : u_2) \in \mathbb{P}^2_{\mathbb{C}} \mid \det(u_0 \mathbb{1} + u_1 F_1 + u_2 F_2) = 0\}$$

with d-by-d identity matrix $\mathbb{1}$

and its dual curve

$$V^*_{F_1,F_2} \subset \mathbb{P}^{2\,*}_{\mathbb{C}}$$

closure of the set of tangent lines at smooth points of V

the boundary generating curve of F_1, F_2 is

$$V^*_{F_1,F_2}(\mathbb{R}) = \{(x_1, x_2) \in \mathbb{R}^2 \mid (1 : x_1 : x_2) \in V^*_{F_1,F_2}\} \subset \mathbb{R}^2$$

Theorem (Kippenhahn)

W_{F_1,F_2} is the convex hull of $V^*_{F_1,F_2}(\mathbb{R})$.

Mathematische Nachr. 6 (1951), 193

Classification of Numerical Ranges ($k = 2$)

$d = 2$, the numerical range W_{F_1,F_2} is an ellipse (possibly degenerate)

$d = 3$, Kippenhahn (1951) derived a classification of W_{F_1,F_2} from the boundary generating curve $V^*_{F_1,F_2}(\mathbb{R})$,
see also Keeler et al. LAA 252 (1997), 115

$d = 4$, Chien and Nakazato derived a classification of W_{F_1,F_2} from $V^*_{F_1,F_2}(\mathbb{R})$, Electronic J. Lin. Alg. 23 (2012), 755

Definition. 3-by-3 matrices $F_1, \ldots, F_k$ are unitarily reducible (otherwise unitarily irreducible) if there is a unitary matrix U such that $U^* F_1 U, \ldots, U^* F_k U$ are of direct sum form

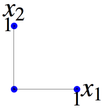
$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}.$$

Numerical Ranges, $d = 3$, Unitarily Reducible

Drawings: boundary generating curves $V^*_{F_1,F_2}(\mathbb{R})$ (blue)

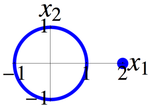
1) $V^*_{F_1,F_2}(\mathbb{R})$ consists of three points

e.g. $F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$



2) $V^*_{F_1,F_2}(\mathbb{R})$ is the union of an ellipse and a point

e.g. $F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

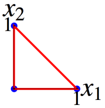


Numerical Ranges, $d = 3$, Unitarily Reducible

Drawings: boundaries of the numerical ranges W_{F_1,F_2} (red)

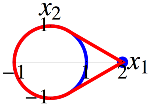
1) W_{F_1,F_2} is a triangle

e.g. $F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$



2) W_{F_1,F_2} is the convex hull of an ellipse and a point

e.g. $F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

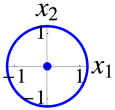


Numerical Ranges, $d = 3$, Unitarily Irreducible

Drawings: boundary generating curves $V^*_{F_1,F_2}(\mathbb{R})$ (blue)

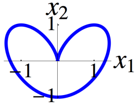
1) $V^*_{F_1,F_2}(\mathbb{R})$ is the union of an ellipse and a point inside

e.g. $F_1 = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & -\mathrm{i} & -\frac{\mathrm{i}}{2} \\ \mathrm{i} & 0 & \frac{\mathrm{i}}{2} \\ \frac{\mathrm{i}}{2} & -\frac{\mathrm{i}}{2} & 0 \end{pmatrix}$



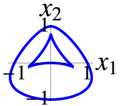
2) $V^*_{F_1,F_2}(\mathbb{R})$ is a quartic curve

e.g. $F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$



3) $V^*_{F_1,F_2}(\mathbb{R})$ is a sextic curve

e.g. $F_1 = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

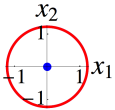


Numerical Ranges, $d = 3$, Unitarily Irreducible

Drawings: boundaries of the numerical ranges W_{F_1,F_2} (red)

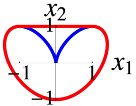
1) W_{F_1,F_2} is an ellipse

e.g. $F_1 = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & -i & -\frac{i}{2} \\ i & 0 & \frac{i}{2} \\ \frac{i}{2} & -\frac{i}{2} & 0 \end{pmatrix}$



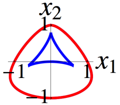
2) W_{F_1,F_2} is the convex hull of a quartic curve

e.g. $F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$



3) W_{F_1,F_2} is the convex hull of a sextic curve

e.g. $F_1 = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$



Problems with Three-Dimensional Joint Numerical Ranges

Boundary generating surface ($k = 3$)

consider the hypersurface

$$V_{F_1,F_2,F_3} = \{u \in \mathbb{P}^3_{\mathbb{C}} : \det(u_0 \mathbb{1} + u_1 F_1 + \cdots + u_3 F_3) = 0\}$$

and its dual variety

$$V^*_{F_1,F_2,F_3} \subset \mathbb{P}^{3\,*}_{\mathbb{C}}$$

closure of the set of tangent planes at smooth points of V

the boundary generating surface of F_1, F_2, F_3 is

$$V^*_{F_1,F_2,F_3}(\mathbb{R}) = \{x \in \mathbb{R}^3 \mid (1 : x_1 : x_2 : x_3) \in V^*_{F_1,F_2,F_3}\} \subset \mathbb{R}^2$$

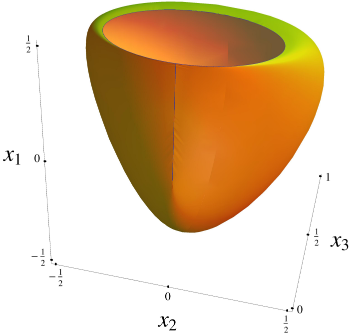
Observation (Chien and Nakazato, LAA 432 (2010), 173)
$V^*_{F_1,F_2,F_3}(\mathbb{R})$ can contain lines, hence $V^*_F(\mathbb{R}) \subset W_F$ is impossible and $\operatorname{conv}(V^*_F(\mathbb{R})) = W_F$ fails.

Example 1

$$F_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ F_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

boundary generating surface

$$V^*_{F_1,F_2,F_3}(\mathbb{R})$$
$$= \{x \in \mathbb{R}^3 \mid -4x_1^2x_3^2 - 4x_2^2x_3^2 + 4x_3^3 - 4x_3^4 + 4x_1x_2^2x_3 - x_2^4 = 0\}$$



Depicted surface:

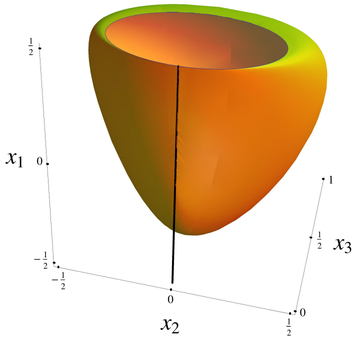
Intersection of $V^*_{F_1,F_2,F_3}(\mathbb{R})$ with the boundary of W_{F_1,F_2,F_3}

Example 1

$$F_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ F_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

boundary generating surface

$$V^*_{F_1,F_2,F_3}(\mathbb{R})$$
$$= \{x \in \mathbb{R}^3 \mid -4x_1^2x_3^2 - 4x_2^2x_3^2 + 4x_3^3 - 4x_3^4 + 4x_1x_2^2x_3 - x_2^4 = 0\}$$



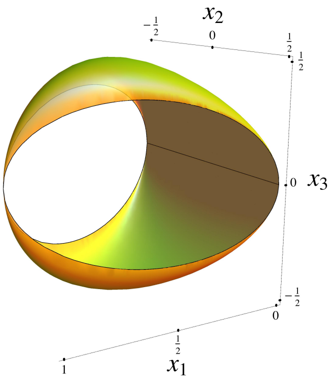
the x_1-axis lies in $V^*_{F_1,F_2,F_3}(\mathbb{R})$

Example 2

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ F_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ F_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

boundary generating surface

$$V^*_{F_1,F_2,F_3}(\mathbb{R}) = \{x \in \mathbb{R}^3 \mid -x_1^2x_2^2 + x_1x_3^2 - x_1^2x_3^2 - x_3^4 = 0\}$$



Depicted surface:

Intersection of $V^*_{F_1,F_2,F_3}(\mathbb{R})$

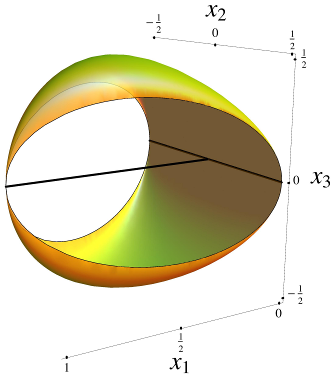
with the boundary of W_{F_1,F_2,F_3}

Example 2

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; F_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; F_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

boundary generating surface

$$V^*_{F_1,F_2,F_3}(\mathbb{R}) = \{x \in \mathbb{R}^3 \mid -x_1^2 x_2^2 + x_1 x_3^2 - x_1^2 x_3^2 - x_3^4 = 0\}$$



the x_1- and x_2-axes

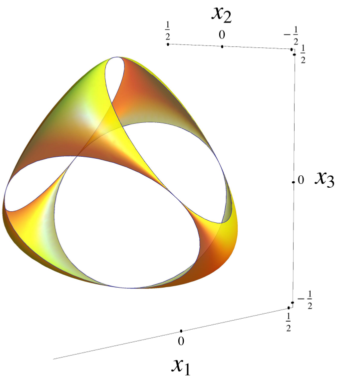
lie in $V^*_{F_1,F_2,F_3}(\mathbb{R})$

Example 3

$$F_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ F_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ F_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

boundary generating surface = *Roman surface*

$$V^*_{F_1,F_2,F_3}(\mathbb{R}) = \{x \in \mathbb{R}^3 \mid x_1 x_2 x_3 - x_1^2 x_2^2 - x_1^2 x_3^2 - x_2^2 x_3^2 = 0\}$$



Depicted surface:

Intersection of $V^*_{F_1,F_2,F_3}(\mathbb{R})$

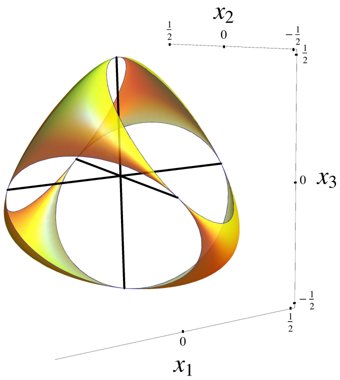
with the boundary of W_{F_1,F_2,F_3}

Example 3

$$F_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ F_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ F_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

boundary generating surface = *Roman surface*

$$V^*_{F_1,F_2,F_3}(\mathbb{R}) = \{x \in \mathbb{R}^3 \mid x_1 x_2 x_3 - x_1^2 x_2^2 - x_1^2 x_3^2 - x_2^2 x_3^2 = 0\}$$



all three coordinate axes

lie in $V^*_{F_1,F_2,F_3}(\mathbb{R})$

Classification of JNRs: State of the Art

Kippenhahn's assertion does not generalize from $k = 2$ to $k = 3$, an algebraic geometry approach seems unavailable !

very little is known about $W_F = W_{F_1,\dots,F_k}$, $k \geq 3$, except for

- corner points (conical points) imply F unitarily reducible

  Binding and Li, LAA 151 (1991), 157

- ovals and reconstruction of F from W_F

  Krupnik and Spitkovsky, LAA 419 (2006), 569

- a maximum of 4 ellipses on the boundary of W_F if $k = d = 3$

  Chien and Nakazato, LAA 430 (2009), 204

Our Approach: Study configurations of exposed faces on the boundary of W_F.

Solution: Graph Embedding

Exposed Faces

an exposed face of a convex set $C \subset \mathbb{R}^n$ is the set of maximizers of a linear functional,

$$\mathbb{F}_C(u) = \operatorname{argmax}\{\langle x, u\rangle : x \in C\}, \qquad u \in \mathbb{R}^n,$$

or the empty set; let $F(u) = u_1 F_1 + \cdots + u_k F_k$, $u \in \mathbb{R}^k$, and

$$\mathbb{E} : \mathcal{M}(M_d) \to W_F,\ \rho \mapsto (\operatorname{tr}(\rho F_1), \ldots, \operatorname{tr}(\rho F_k));$$

then

$$\mathbb{E}^{-1}\left(\mathbb{F}_{W_F}(u)\right) = \mathbb{F}_{\mathcal{M}(M_d)}(F(u))$$

and

$$\mathbb{F}_{\mathcal{M}(M_d)}(F(u)) = \mathcal{M}(pM_d p)$$

where p is the spectral projection of $F(u)$ corresponding to the maximal eigenvalue

Large Faces

we assume $k = d = 3$ and call large face an exposed face of W_F which is neither $\varnothing$, nor a singleton, nor equal to all of W_F

Lemma (Szymański, SW, Życzkowski)

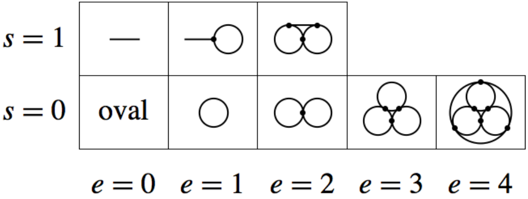
1) Every large face is a segment or a filled ellipse.
2) Each two distinct large faces intersect in a singleton.
3) If G_1, G_2, G_3 are mutually distinct large faces and $G_1 \cap G_2 \cap G_3 = \varnothing$, then W_F has a corner point.
4) If there are two distinct large faces which are segments, then W_F has a corner point.

Graph Embedding

2) and 3) of the lemma show that a complete graph K_n embeds into the union of large faces with one vertex on each large face

the boundary of W_F is homeomorphic to the sphere S^2 so $n \leq 4$

Theorem (Szymański, SW, Życzkowski)
Let $k = d = 3$. If W_F has no corner point, then the set of large faces has one of the following configurations.



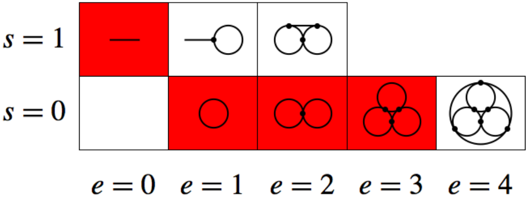
Finding Examples

Finding Candidates

searching for candidates belonging to each class, we used

- random matrices
- guessing

and found some new examples (red)



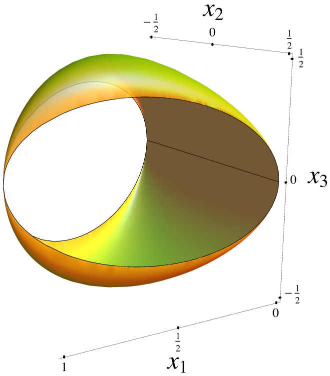
Question: How to determine the class of an example?

Large Faces of Example 2

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

the exposed faces with normal vectors $u = (-1, 0, 0), (1, \pm 2, 0)$ are large faces as the maximal eigenvalues are degenerate:

$$F(-1,0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad F(1,\pm 2,0) = \begin{pmatrix} 0 & \pm 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



how can we be sure there are no further large faces?

by finding all degenerate eigenvalues in the hermitian pencil spanned by F_1, F_2, F_3

Discriminant as a Sum of Squares

the discriminant of the polynomial $p(\lambda) = -\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ is

$$\begin{aligned}\mathrm{Disc}_\lambda(p) &= -(27a_3^2 + 18a_1a_2a_3 - 4a_1^3a_3 + 4a_2^3 - a_1^2a_2^2) \\ &= (\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2\end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the roots of p

the discriminant of $A \in M_d$ is $\mathrm{Disc}(A) = \mathrm{Disc}_\lambda(\det(A - \lambda\mathbb{1}))$

Theorem (Ilyushechkin)
Let $A \in M_d$ be a normal matrix. Let A_* be the $d^2 \times d$-matrix with columns the coefficients of $\mathbb{1}, A, A^2, \ldots, A^{d-1}$, all in the same order. Let M_ν denote the ν-minor of A_*. Then

$$|\mathrm{Disc}(A)| = \sum_{\nu\subset\{1,\ldots,d\}^2, |\nu|=d} |M_\nu|^2.$$

Mathematical Notes 51 (1992), 230

Squared Minors of $F(u_1, u_2, u_3)$ from Example 2

$$\Big\{\frac{1}{16}\,u1^2\,u2^2\,u3^2,\ 0,\ \frac{u2^2\,u3^4}{64},\ \frac{u2^4\,u3^2}{64},\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ \frac{u2^4\,u3^2}{64},\ \frac{1}{64}\left(-4\,u1^2\,u2+u2^3\right)^2,$$

$$\frac{1}{16}\,u1^2\,u2^2\,u3^2,\ \frac{u3^6}{64},\ \frac{u2^2\,u3^4}{64},\ 0,\ \frac{u2^2\,u3^4}{64},\ \frac{u2^4\,u3^2}{64},\ \frac{u2^2\,u3^4}{64},\ \frac{u2^4\,u3^2}{64},\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,$$

$$\frac{u2^4\,u3^2}{64},\ \frac{1}{64}\left(-4\,u1^2\,u2+u2^3\right)^2,\ 0,\ \frac{u3^6}{64},\ 0,\ \frac{u1^2\,u3^4}{16},\ \frac{u2^2\,u3^4}{64},\ 0,\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,$$

$$\frac{u2^2\,u3^4}{64},\ \frac{u2^4\,u3^2}{64},\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ 0,\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ 0,\ 0,\ 0,\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ 0,\ 0,\ 0,$$

$$0,\ 0,\ \frac{u2^4\,u3^2}{64},\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ \frac{u2^4\,u3^2}{64},\ \frac{1}{64}\,u2^2\left(4\,u1^2-u2^2+u3^2\right)^2,\ 0,\ 0,\ \frac{u2^4\,u3^2}{64},\ 0,$$

$$\frac{1}{16}\,u1^2\,u2^2\,u3^2,\ \frac{u2^4\,u3^2}{64},\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ 0,\ 0,\ 0,\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ \frac{u2^2\,u3^4}{64},\ 0,\ \frac{u2^2\,u3^4}{64},$$

$$\frac{1}{64}\left(-u2^2\,u3+u3^3\right)^2,\ 0,\ 0,\ \frac{u2^2\,u3^4}{64},\ 0,\ 0,\ \frac{u2^2\,u3^4}{64},\ \frac{u2^4\,u3^2}{64},\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ \frac{u2^4\,u3^2}{64},$$

$$\frac{1}{64}\,u2^2\left(4\,u1^2-u2^2+u3^2\right)^2,\ 0,\ 0,\ \frac{u2^4\,u3^2}{64},\ 0,\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ \frac{u2^4\,u3^2}{64},\ \frac{u2^2\,u3^4}{64},\ 0,$$

$$\frac{1}{16}\,u1^2\,u2^2\,u3^2,\ \frac{u2^2\,u3^4}{64},\ \frac{1}{64}\left(-u2^2\,u3+u3^3\right)^2,\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ 0,\ \frac{u2^2\,u3^4}{64},\ 0,\ \frac{u2^2\,u3^4}{64}\Big\}$$

Squared Minors of $F(u_1, u_2, u_3)$ from Example 2

$$\left\{\frac{1}{16}\,u1^{2}\,u2^{2}\,u3^{2},\ 0,\ \frac{u2^{2}\,u3^{4}}{64},\ \frac{u2^{4}\,u3^{2}}{64},\ \frac{1}{16}\,u1^{2}\,u2^{2}\,u3^{2},\ \frac{u2^{4}\,u3^{2}}{64},\ \frac{1}{64}\left(-4\,u1^{2}\,u2+u2^{3}\right)^{2},\right.$$

$$\frac{1}{16}\,u1^{2}\,u2^{2}\,u3^{2},\ \frac{u3^{6}}{64},\ \frac{u2^{2}\,u3^{4}}{64},\ 0,\ \frac{u2^{2}\,u3^{4}}{64},\ \frac{u2^{4}\,u3^{2}}{64},\ \frac{u2^{2}\,u3^{4}}{64},\ \frac{u2^{4}\,u3^{2}}{64},\ \frac{1}{16}\,u1^{2}\,u2^{2}\,u3^{2},$$

$$\frac{u2^{4}\,u3^{2}}{64},\ \frac{1}{64}\left(-4\,u1^{2}\,u2+u2^{3}\right)^{2},\ 0,\ \frac{u3^{6}}{64},\ 0,\ \frac{u1^{2}\,u3^{4}}{16},\ \frac{u2^{2}\,u3^{4}}{64},\ 0,\ \frac{1}{16}\,u1^{2}\,u2^{2}\,u3^{2},$$

$$\frac{u2^{2}\,u3^{4}}{64},\ \frac{u2^{4}\,u3^{2}}{64},\ \frac{1}{16}\,u1^{2}\,u2^{2}\,u3^{2},\ 0,\ \frac{1}{16}\,u1^{2}\,u2^{2}\,u3^{2},\ 0,\ 0,\ 0,\ \frac{1}{16}\,u1^{2}\,u2^{2}\,u3^{2},\ 0,\ 0,\ 0,$$

$$0,\ 0,\ \frac{u2^{4}\,u3^{2}}{64},\ \frac{1}{16}\,u1^{2}\,u2^{2}\,u3^{2},\ \frac{u2^{4}\,u3^{2}}{64},\ \frac{1}{64}\,u2^{2}\left(4\,u1^{2}-u2^{2}+u3^{2}\right)^{2},\ 0,\ 0,\ \frac{u2^{4}\,u3^{2}}{64},\ 0,$$

$$\frac{1}{16}\,u1^{2}\,u2^{2}\,u3^{2},\ \frac{u2^{4}\,u3^{2}}{64},\ \frac{1}{16}\,u1^{2}\,u2^{2}\,u3^{2},\ 0,\ 0,\ 0,\ \frac{1}{16}\,u1^{2}\,u2^{2}\,u3^{2},\ \frac{u2^{2}\,u3^{4}}{64},\ 0,\ \frac{u2^{2}\,u3^{4}}{64},$$

$$\frac{1}{64}\left(-u2^{2}\,u3+u3^{3}\right)^{2},\ 0,\ 0,\ \frac{u2^{2}\,u3^{4}}{64},\ 0,\ 0,\ \frac{u2^{2}\,u3^{4}}{64},\ \frac{u2^{4}\,u3^{2}}{64},\ \frac{1}{16}\,u1^{2}\,u2^{2}\,u3^{2},\ \frac{u2^{4}\,u3^{2}}{64},$$

$$\frac{1}{64}\,u2^{2}\left(4\,u1^{2}-u2^{2}+u3^{2}\right)^{2},\ 0,\ 0,\ \frac{u2^{4}\,u3^{2}}{64},\ 0,\ \frac{1}{16}\,u1^{2}\,u2^{2}\,u3^{2},\ \frac{u2^{4}\,u3^{2}}{64},\ \frac{u2^{2}\,u3^{4}}{64},\ 0,$$

$$\left.\frac{1}{16}\,u1^{2}\,u2^{2}\,u3^{2},\ \frac{u2^{2}\,u3^{4}}{64},\ \frac{1}{64}\left(-u2^{2}\,u3+u3^{3}\right)^{2},\ \frac{1}{16}\,u1^{2}\,u2^{2}\,u3^{2},\ 0,\ \frac{u2^{2}\,u3^{4}}{64},\ 0,\ \frac{u2^{2}\,u3^{4}}{64}\right\}$$

Evaluation of Squared Minors from Example 2

- if the exposed face $\mathbb{F}_{W_F}(u)$ is a large face then $u_3 = 0$ because

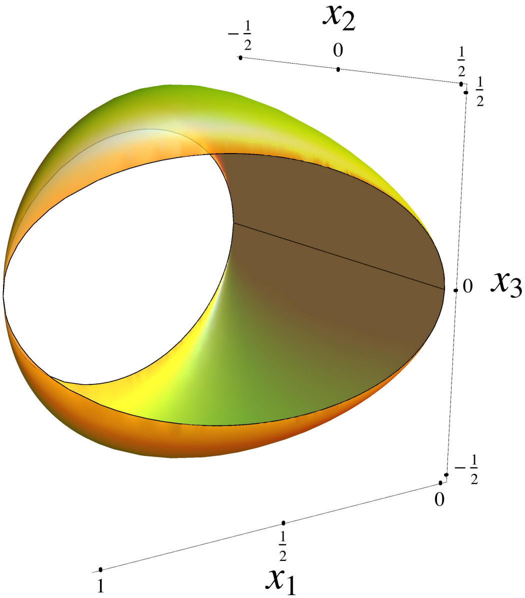
$$u_3 \neq 0 \quad \Rightarrow \quad |\mathrm{Disc}(F(u))| \geq |M_{\{(1,1),(1,3),(2,2)\}}|^2 = \frac{u_3^6}{64} > 0$$

- if the exposed face $\mathbb{F}_{W_F}(u_1, u_2, 0)$ is a large face then $u_2 = 0$ or $u_2 = \pm 2u_1$ since

$$|\mathrm{Disc}(F(u))| \geq |M_{\{(1,1),(1,2),(3,3)\}}|^2 = \frac{u_2^2(u_2^2 - 4u_1^2)^2}{64}$$

Result: Example 2 has exactly three large faces, the segment $\mathbb{F}_{W_F}(-1, 0, 0)$ and the two ellipses $\mathbb{F}_{W_F}(1, \pm 2, 0)$.

All Large Faces of Example 2



$\mathbb{F}_{W_F}(-1, 0, 0)$

$\mathbb{F}_{W_F}(1, \pm 2, 0)$

Conclusion

Summary:

We have a classification of joint numerical ranges in the simplest three-dimensional case of $k = d = 3$.

Questions:

- Can we find classifications of L_F for $k > 3$, $d = 3$?
  probably yes, but graph embedding into S^{k-1} is no constraint any more
- Can we find a classification of W_F for $k = 3$, $d = 4$?
  unclear, even determining large faces is very hard, as $\mathrm{Disc}(F(u))$ is a sum of $\binom{16}{4} = 1820$ squares

Reference:

Konrad Szymański, SW, Karol Życzkowski, *Classification of joint numerical ranges of three hermitian matrices of size three*, LAA 545 (2018), 148

Thank you