Choquet's Theorem for Constrained Sets of Quantum States

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Abstract

Models of realistic quantum communication need an energy bound at the source of a communication channel. A common choice is a bound on the expected value of an unbounded energy operator. Although this constraint has been applied successfully, there are basic questions of functional analysis still open. Here, we prove a version of Choquet's theorem, which asserts that every state with bounded energy is the barycenter of a probability measure supported by pure sates with bounded energy. This result is an important step forward in the functional analysis of constrained states, as it essentially simplifies definitions of several characteristics used in quantum information theory.

References.

[1] S. W. Weis and M. E. Shirokov, *Extreme points of the set of quantum states with bounded energy*, Russ. Math. Surv. **76**:1, 190–192 (2021).

[2] S. Weis and M. Shirokov, *The face generated by a point, generalized affine constraints, and quantum theory*, Journal of Convex Analysis **28**:3, 847–870 (2021).

Quantum Technology — Information Theory

Direct communication through optical fiber

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Information theory helps to optimize communication tasks:

Energy-Constraints

In order to quantify the capacity of a communication channel, one has to take into account a tradeoff between the energy expended and the communication achieved.

- T. M. Cover and J. A. Thomas, Elements of Information Theory, 1991
- A. S. Holevo, *On quantum communication channels with constrained inputs*, `arXiv:quant-ph/9705054` (1997)

Quantum States, Channels, Entropy

Let $\mathcal{H}$ be a separable Hilbert space and $\mathfrak{T}(\mathcal{H})$ the Banach space of trace-class operators.

States

In quantum information theory, a state is an element of the convex set of density operators

$$\mathfrak{S}(\mathcal{H}) = \{\rho \in \mathfrak{T}(\mathcal{H}) \mid \rho \succeq 0, \mathrm{Tr}(\rho) = 1\}.$$

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Channels

By channel $\Phi : A \to B$ we mean a linear, bounded, trace-preserving, completely positive map $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$. Completely positive means that the map $\Phi \otimes \mathrm{Id}_n$ is positive for all $n = 1, 2, \ldots$.

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Entropy

The von Neumann entropy $H(\rho) \in [0, +\infty]$ of a state $\rho \in \mathfrak{S}(\mathcal{H})$ is the number $H(\rho) = -\mathrm{Tr}[\rho \log(\rho)]$.

The Quantum Capacity of a Constrained Channel

The quantum capacity $Q(\Phi, F, E)$ of the constrained channel $\Phi : A \to B$ is the asymptotically optimal rate at which quantum information can be send if the input states ρ to the composite channel $\Phi^{\otimes n}$ satisfy $\mathrm{Tr}(\rho F^{(n)}) \leq nE$ with energy observable $F^{(n)} = F \otimes I \otimes \cdots \otimes I + I \otimes F \otimes \cdots \otimes I + \ldots + I \otimes \cdots \otimes I \otimes F$.

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Theorem (Wilde and Qi '18, arXiv:1609.01997)

Let $\mathrm{Tr}\, e^{-\theta F} < \infty$ for all $\theta > 0$ (Gibbs hypothesis), and let

$$\sup_{\rho \in \mathfrak{S}(\mathcal{H}_A): \mathrm{Tr}\, \rho F \leq E} H[\Phi(\rho)] < \infty. \qquad \text{(finite output entropy)}$$

Then the quantum capacity of the constrained channel Φ is

$$Q(\Phi, F, E) = \lim_{n \to \infty} \frac{1}{n} I_{\mathrm{c}}(\Phi^{\otimes n}, F^{(n)}, nE).$$

$I_{\mathrm{c}}(\Phi, F, E) = \sup_{\rho \in \mathfrak{S}(\mathcal{H}_A): \mathrm{Tr}\, \rho F \leq E} \left[H(B)_\omega - H(RB)_\omega \right]$ (coherent information)

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- Uniform continuity bounds: Winter arXiv:1712.10267, Shirokov arXiv:1706.00361, Becker and Datta arXiv:1810.00863

Generalized Compactness

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μ-Compactness

Let X be a closed, bounded subset of a separable Banach space and $M(X)$ the set of Borel probability measures on X; the barycenter of $\mu \in M(X)$ is

$$b(\mu) = \int_X x \, d\mu(x);$$

X is μ-compact if the pre-image under $b : M(X) \to \overline{\mathrm{conv}}(X)$ is compact for every compact subset of $\overline{\mathrm{conv}}(X)$.

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Theorem 1 (Protasov and Shirokov '10, arXiv:1002.3610)

Let C be a closed, bounded, μ-compact, convex set, and a separable metric space. Then

$$C = \overline{\mathrm{conv}}(\mathrm{ext}\, C), \qquad \text{(Krein-Milman theorem)}$$

$$C = b\big(M(\overline{\mathrm{ext}\, C})\big). \qquad \text{(Choquet theorem)}$$

Generalized Affine Constraints

Generalized Affine Maps

In the sequel, let V be a real vector space and $K \subseteq V$ a convex set. A generalized affine map on K is a map $f : K \to \mathbb{R} \cup \{+\infty\}$ that satisfies

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y). \qquad x, y \in K, \quad \lambda, \mu \geq 0, \quad \lambda + \mu = 1.$$

Let $\ell \in \mathbb{N}$, let $f_1, f_2, \ldots, f_\ell$ be generalized affine maps on K, and let $\alpha_1, \alpha_2, \ldots, \alpha_\ell \in \mathbb{R}$. We define the sublevel set

$$K_\ell = \{x \in K : f_k(x) \leq \alpha_k \ \forall k = 1, \ldots, \ell\}.$$

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Expected Value Functional

Let H be an arbitrary positive operator on $\mathcal{H}$. Let $P_n = \int_0^n dE_H(\lambda)$ be the spectral projector of H corresponding to $[0, n]$, where E_H is the spectral measure of H. The expected value functional of H is the map defined by

$$\mathfrak{S}(\mathcal{H}) \to [0, +\infty], \qquad \rho \mapsto \operatorname{Tr} \rho H = \lim_{n\to\infty} \operatorname{Tr}(\rho H P_n).$$

Does the Sublevel Set $\mathfrak{S}(\mathcal{H})_\ell$ Have Mixed Extreme Points?

Consider several expected value functionals $f_1, f_2, \ldots, f_\ell : \mathfrak{S}(\mathcal{H}) \to [0, \infty]$.

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• The sublevel set $\mathfrak{S}(\mathcal{H})_\ell$ is closed, as $f_1, f_2, \ldots, f_\ell$ are lower semi-continuous. Theorem 1 shows that any state in $\mathfrak{S}(\mathcal{H})_\ell$ is the barycenter of some Borel probability measure supported by $\overline{\mathrm{ext}(\mathfrak{S}(\mathcal{H})_\ell)}$.

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• This measure would be supported by the closed set of pure states

$$\mathrm{ext}(\mathfrak{S}(\mathcal{H})_\ell) = \mathfrak{S}(\mathcal{H})_\ell \cap \mathrm{ext}\,\mathfrak{S}(\mathcal{H}),$$

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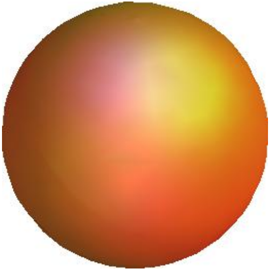
Question

Under which conditions is every extreme point of $\mathfrak{S}(\mathcal{H})_\ell$ a pure state?

Affine Constraints on the Bloch Ball $\mathfrak{S}(\mathbb{C}^2)$



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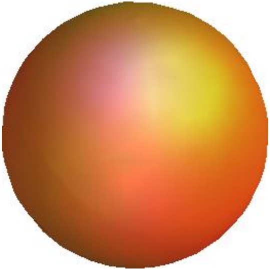

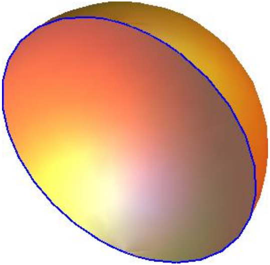

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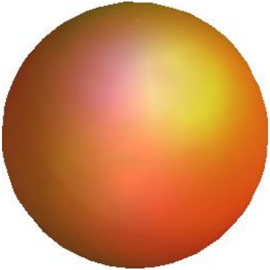

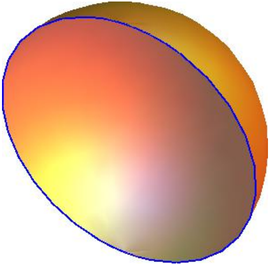

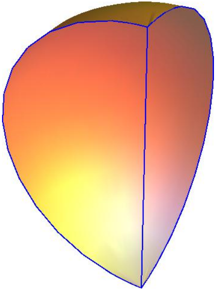

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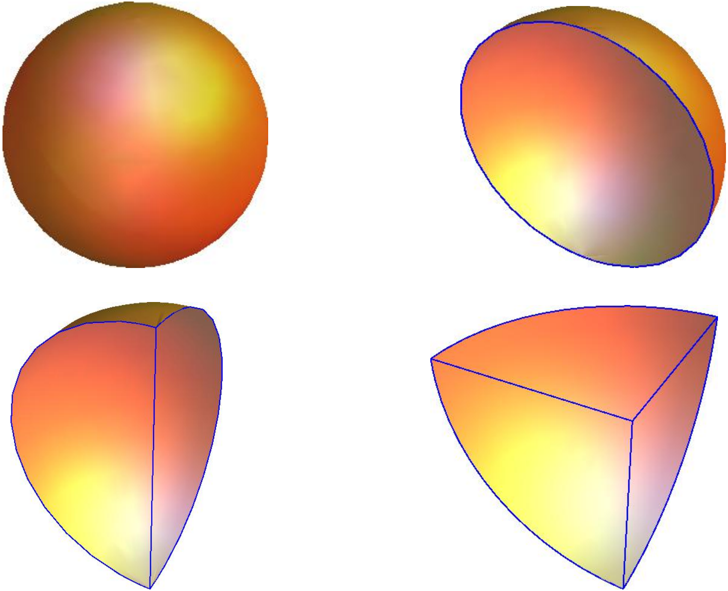

The Face Generated by a Point

Problem: ∞-dim. convex sets may lack "interior points" (A. Barvinok, *A Course in Convexity*, Providence, R.I: AMS, 2002). We focus on faces generated by points.

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Faces

A subset $E \subseteq K$ is an extreme subset of K if whenever $x = (1 - \lambda)y + \lambda z$ lies in E for some $y, z \in K$ and $\lambda \in (0, 1)$, then y and z are also in E. A face of K is a convex, extreme subset of K. The face of K generated by $x \in K$, denoted $F_K(x)$, is the intersection of all faces of K containing x.

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Theorem (W., Shirokov '20, arXiv:2003.14302)

Let V be a real vector space, $K \subseteq V$ a convex subset, and $x \in K$. Then

$$\operatorname{aff} F_K(x) = \big\{ y \in V \mid \exists \epsilon > 0 : x \pm \epsilon(y - x) \in F_K(x) \big\}.$$

In particular, x lies in the "interior" of $F_K(x)$.

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Proof. "$\supseteq$": Every open segment in $F_K(x)$ extends to a line in aff $F_K(x)$.
We prove "$\subseteq$" using the Kuratowski-Zorn lemma.

Gaps in the Face Dimensions

If the convex set K is replaced with its sublevel set K_1, then the dimension of the face generated by a point may decrease by at most one.

Theorem

Let x be a point in K_1. If the face $F_{K_1}(x)$ of K_1 generated by x has dimension $m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, then the face $F_K(x)$ of K generated by x has dimension m or $m + 1$.

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Iterating the theorem, we exploit gaps in the face dimensions of K.

Corollary

Let K have no face with dimension $1, \ldots, \ell$. Then every extreme point of the sublevel set K_ℓ is an extreme point of K.

The Gap Between 0 and 3 in the Face Dimensions of $\mathfrak{S}(\mathcal{H})$

The faces of the set of density operators have dimensions $0, 3, 8, \ldots, n^2-1, \ldots, \infty$.

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Krein-Milman and Choquet Theorem

The set of extreme points $\mathrm{ext}(\mathfrak{S}(\mathcal{H})_2)$ is closed and is equal to the set of pure states in $\mathfrak{S}(\mathcal{H})_2$.

- The set $\mathfrak{S}(\mathcal{H})_2$ is the closure of the convex hull of $\mathrm{ext}(\mathfrak{S}(\mathcal{H})_2)$.
- Any state in $\mathfrak{S}(\mathcal{H})_2$ is the barycenter of some Borel probability measure supported by $\mathrm{ext}(\mathfrak{S}(\mathcal{H})_2)$.

Maximizing Convex Functions on the Sublevel Set $\mathfrak{S}(\mathcal{H})_2$

Corollary

- Let $f : \mathfrak{S}(\mathcal{H})_2 \to [-\infty, \infty]$ be a convex function. If f is either lower semicontinuous or upper semicontinuous and upper bounded, then

$$\sup\{f(\rho) : \rho \in \mathfrak{S}(\mathcal{H})_2\} = \sup\{f(\rho) : \rho \in \mathrm{ext}(\mathfrak{S}(\mathcal{H})_2)\}. \tag{1}$$

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- If f is upper semicontinuous and one of the operators defining the functionals $f_i : \rho \mapsto \mathrm{Tr}\,\rho H_i$, $i = 1, 2$, has a discrete spectrum of finite multiplicity, then $\mathfrak{S}(\mathcal{H})_2$ is compact and the supremuma in (1) are attained.

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See the Refs. [1,2] for applications of these results to the Minimal Output Entropy and to the Operator E-Norms.

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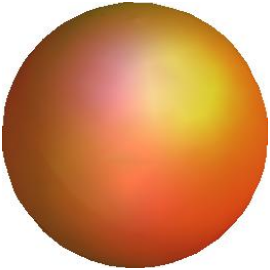

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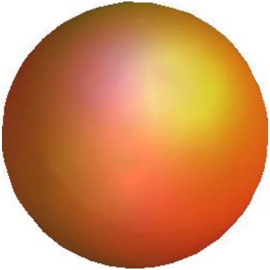

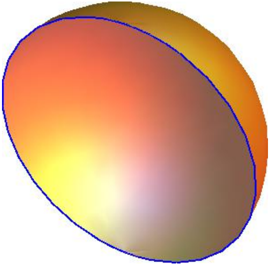

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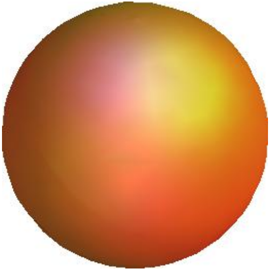

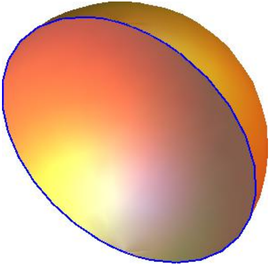

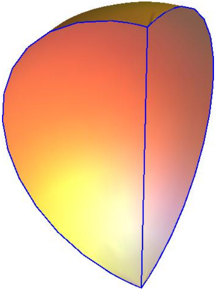

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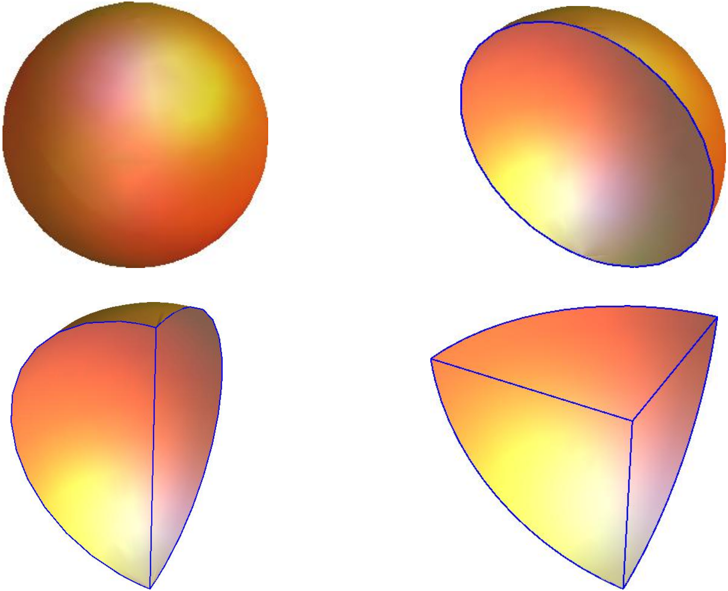

Thank you for your attention!

Appendix with Applications

Applications: Minimal Output Entropy I

Minimal Output Entropy

The minimal output entropy of a channel $\Phi : A \to B$ is defined as

$$H_{\min}(\Phi) = \inf_{\rho \in \mathfrak{S}(\mathcal{H}_A)} H(\Phi(\rho)) = \inf_{\varphi \in \mathcal{H}_{A,\mathbf{1}}} H(\Phi(|\varphi\rangle\langle\varphi|)),$$

where $\mathcal{H}_{A,1}$ is the unit sphere in $\mathcal{H}_A$.

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where $\mathcal{H}_{A,1}$ is the unit sphere in $\mathcal{H}_A$.

The additivity of the minimal output entropy, which means that

$$H_{\min}(\Phi \otimes \Psi) = H_{\min}(\Phi) + H_{\min}(\Psi)$$

for all channels Φ and Ψ, was disproved by Hasting ('08, arXiv:0809.3972). This is important, because the additivity of $H_{\min}$ is equivalent to the additivity of the Holevo information (Shor '03, arXiv:quant-ph/0305035). The additivity of the Holevo information would imply that the classical capacity is $C(\Phi) = \chi(\Phi)$, without the regularization $C(\Phi) = \lim_{n\to\infty} \frac{1}{n}\chi(\Phi^{\otimes n})$.

Applications: Minimal Output Entropy II

Constrained Minimal Output Entropy

In studies of an infinite-dimensional channels $\Phi : A \to B$, it is reasonable to consider the constrained minimal output entropy

$$H_{\min}(\Phi, G, E) = \inf_{\rho \in \mathfrak{S}(\mathcal{H}_A) : \operatorname{Tr} \rho G \leq E} H(\Phi(\rho)), \tag{2}$$

where G is a positive operator, the energy observable.

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• If $\dim \mathcal{H}_A < \infty$, then the infimum in (2) can be taken only over pure states satisfying the condition $\operatorname{Tr} \rho G \leq E$ (Memarzadeh, Mancini '16, arXiv:1605.04525). Our results prove the analogous assertion for an arbitrary ∞-dim. channel Φ and for any energy observable G.

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• $H_{\min}(\hat{\Phi}, G, E) = H_{\min}(\Phi, G, E)$ holds for every complementary channel $\hat{\Phi}$, as $H[\hat{\Phi}(\rho)] = H[\Phi(\rho)]$ holds for all pure states (A. S. Holevo, *Quantum Systems, Channels, Information*, Berlin: De Gruyter, 2019).

Applications: Operator E-Norms I

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Stinespring Theorem

Given a completely positive linear map $\Phi : A \to B$, there exists a Hilbert space $\mathcal{H}_R$ and an operator $V_\Phi : \mathcal{H}_A \to \mathcal{H}_{RB}$ such that $\Phi(\rho) = \mathrm{Tr}_R\, V_\Phi \rho V_\phi^*$ for all $\rho \in \mathfrak{T}(\mathcal{H}_A)$.

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Let G be a grounded Hamiltonian with dense domain. Given $E > 0$, the operator E-norm of $A \in \mathfrak{B}(\mathcal{H}_A, \mathcal{H}_{RB})$ is

$$\|A\|_E^G \doteq \sup_{\rho\in\mathfrak{S}(\mathcal{H}_A):\mathrm{Tr}\,\rho G\leq E} \sqrt{\mathrm{Tr}\, A\rho A^*}.$$

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Corollary

Our results show that the operator E-norm is a constrained version of the operator norm, that is to say, $\|A\|_E^G = \sup_{\varphi \in \mathcal{H}_A, \langle \varphi | \varphi \rangle = 1, \langle \varphi | G | \varphi \rangle \leq E} \|A\varphi\|$.

Applications: Operator E-Norms II

Theorem (Shirokov '18, arXiv:1806.05668)

For any completely positive linear maps Φ and Ψ from $\mathfrak{T}(\mathcal{H}_A)$ to $\mathfrak{T}(\mathcal{H}_B)$, the following inequalities hold.

$$\frac{\|\Phi-\Psi\|_{\diamond,E}^{G}}{\sqrt{\|\Phi\|_{\diamond,E}^{G}}+\sqrt{\|\Psi\|_{\diamond,E}^{G}}} \leq \inf_{V_\Phi, V_\Psi} \|V_\Phi - V_\Psi\|_E^G \leq \sqrt{\|\Phi-\Psi\|_{\diamond,E}^{G}},$$

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The theorem improves the original KSW-theorem, which uses the unconstrained diamond norm on the space of Hermitian-preserving linear maps and the unconstrained operator norm on the set of Stinespring operators.