Open linear maps and Gibbs varieties

Nonlinear Algebra Seminar
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Czech Technical University in Prague

Abstract

This talk explores the idea that a phase transition of a system occurs at a given state if an arbitrarily small local change of this state forces the global state to change abruptly. Surprisingly, this —apparently too general— idea has a practical meaning in the setting of finite-dimensional quantum systems, as the collection of partial traces is not an open map. This lack of openness contrasts with classical mechanics.

We propose to use Gibbs manifolds and Gibbs varieties as tools to characterize the openness and we discuss examples and prior results. Related discontinuity phenomena of the maximum-entropy inference were associated with phase transitions in the work of Chen et al. `arXiv:1406.5046 [quant-ph]` before.

Matrix Systems

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$$\text{where} \quad \mathcal{R}^* := \{\mathbb{C}\text{-linear maps } \mathcal{R} \to \mathbb{C}\}$$

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"Density Matrices" in *-Algebras

If $\mathcal{R}$ is a *-algebra, then $\mathcal{C}^\vee = \mathcal{C}$ and $\mathcal{D} = \{\rho \in \mathcal{R} : \rho \geq 0, \mathrm{Tr}(\rho) = 1\}$.

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Proof.

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Let $\mathcal{R}_2 \subset \mathcal{R}_1$ be matrix systems in M_n and $\pi : \mathcal{H}_1 \to \mathcal{H}_2$ the orthogonal projection (Frobenius inner product).

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Restricting states means projecting density matrices

The following diagram commutes.

$$\begin{tikzcd} \mathcal{D}_1 \arrow[r, "r_1"] \arrow[d, "\pi"'] & \mathcal{S}_1 \arrow[d, "\varphi \mapsto \varphi|_{\mathcal{R}_2}"] \\ \mathcal{D}_2 \arrow[r, "r_2"] & \mathcal{S}_2 \end{tikzcd}$$

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- Every affine map from a convex set onto a polytope is open.

Lack of openness in the quantum domain

Example

Pauli matrices

$$X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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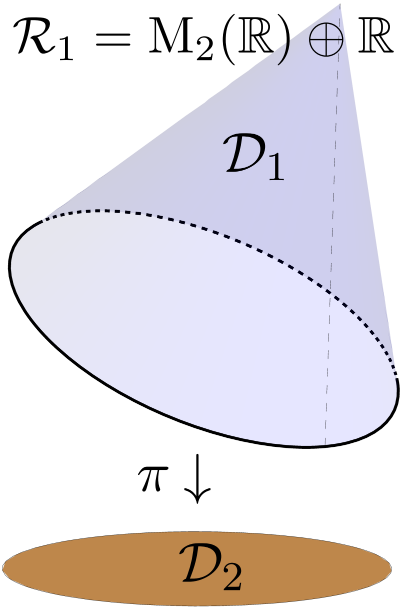

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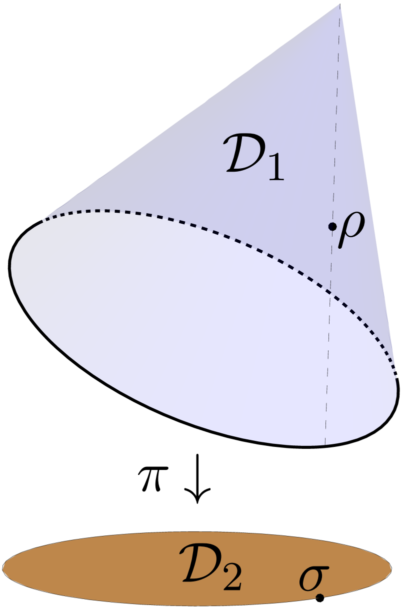

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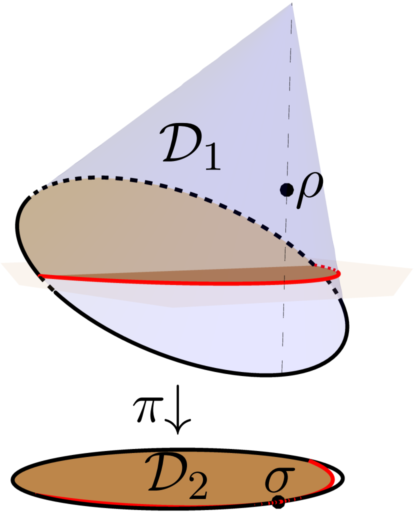

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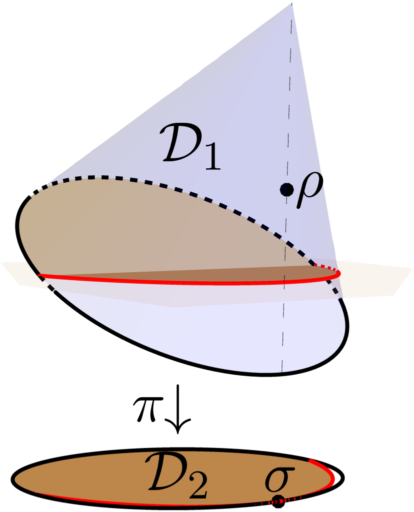

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If π is open at ρ then any sufficiently small change of $\sigma = \pi(\rho)$ is matched by an arbitrarily small change of ρ.

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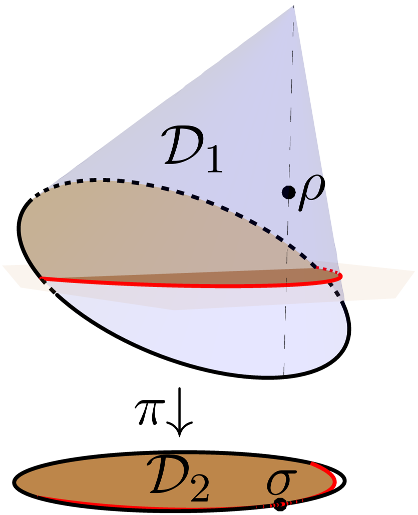

Discontinuity in the quantum domain

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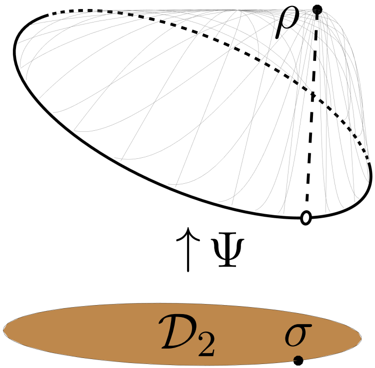

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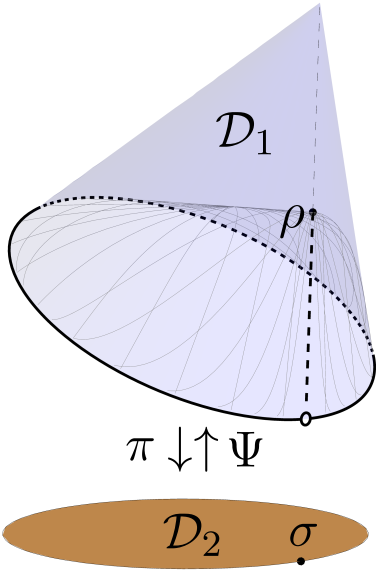

Discontinuity in the quantum domain

The inference map

Let $\mathcal{R}_1 \subset \mathrm{M}_n$ be a *-algebra.

- von Neumann entropy
  $S(\rho) := -\operatorname{Tr} \rho \log(\rho), \ \rho \in \mathcal{D}_1$
- inference map $\Psi : \mathcal{D}_2 \to \mathcal{D}_1$
  $\Psi(\sigma) := \operatorname{argmax}_{\pi(\rho)=\sigma} S(\rho)$.

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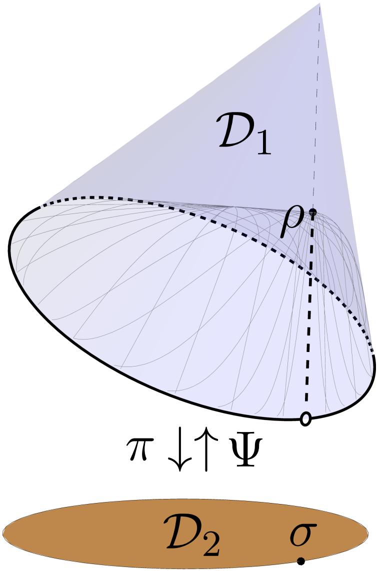
W. (2014)

The inference map Ψ is continuous at $\sigma \in \mathcal{D}_2$ if and only if π is open at $\Psi(\sigma)$.



Many-Party Systems

Local Hamiltonians

The algebra of observables of N qubits is the tensor product $\mathrm{M}_2^{\otimes N}$. A k-local Hamiltonian is a sum of terms $A_1 \otimes \cdots \otimes A_N$ having at most k non-scalar factors $A_i \in \mathcal{H}(\mathrm{M}_2)$, $i = 1, \ldots, N$. Let $\mathcal{L}_k$ be the matrix system generated by the k-local Hamiltonians.

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Partial Traces vs. Projection to $\mathcal{L}_k$

We have $\mathrm{red}_k = \mathrm{red}_k \circ \pi$ and $\mathrm{red}_k : \mathcal{D}(\mathcal{L}_k) \to \mathrm{red}_k(\mathcal{A}_\Omega)$ is an affinity.

Discontinuity for a 3-Qubit System $\mathcal{R}_1 = \mathcal{A}_{\{1,2,3\}}$

Chen et al. (2015)

Let $\mathcal{R}_2 := \mathcal{L}_2$. The inference map $\Psi : \mathcal{D}_2 \to \mathcal{D}_1$ is discontinuous at $\pi(\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)) = \pi(\frac{1}{2}P)$, where $P := |000\rangle\langle 000| + |111\rangle\langle 111|$.

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- $\gamma : [0,1) \to \mathcal{D}_2$, $\epsilon \mapsto \pi(F_-(H_\epsilon))$ is a continuous curve along which $\Psi : \mathcal{D}_2 \to \mathcal{D}_1$ is discontinuous:

$$\Psi(\gamma(\epsilon)) = \begin{cases} P_-(H_\epsilon) & \text{if } \epsilon > 0 \\ \frac{1}{2}P_-(H_0) = \frac{1}{2}P & \text{if } \epsilon = 0 \end{cases} .$$

Lower Semicontinuity of the Face Function of $\mathcal{D}_2$

The setvalued face function $F : \mathcal{D}_2 \to \mathcal{D}_2$ maps $\sigma \in \mathcal{D}_2$ to the union of $\{\sigma\}$ and all closed segments in $\mathcal{D}_2$ whose open segments contain σ.

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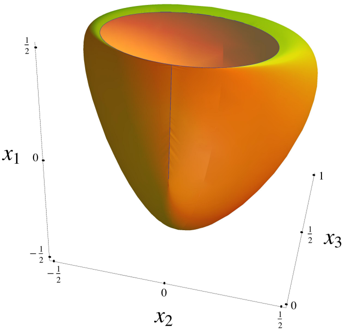

Introducing coordinates on $\mathcal{D}_2$, we define the joint numerical range of $F_1, \ldots, F_l \in \mathcal{H}(\mathrm{M}_n)$ by $W = \{(x_1, \ldots, x_l) : x_i = \langle \rho, F_i \rangle, \rho \in \mathcal{D}(\mathrm{M}_n)\}$.

Note that W is affinely isomorphic to $\mathcal{D}(\mathrm{span}_{\mathbb{C}}(\mathbb{1}_n, F_1, \ldots, F_l))$.

Example ($n = l = 3$), see below for the matrices.

Algebraic Geometry of $\mathcal{D}_2$

Let $F_1, F_2, F_3 \in \mathcal{H}(\mathrm{M}_n)$.

- hyperbolic hypersurface / symmetroid (Ottem et al. 2015)

$$V = \{u \in \mathbb{CP}^3 : \det(u_0 \mathbb{1} + u_1 F_1 + \cdots + u_3 F_3) = 0\}$$

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- V^*_{reg}, the set of regular points in V^*, and the regular part of $V^*(\mathbb{R})$:

$$V^*(\mathbb{R})_{reg} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (1 : x_1 : x_2 : x_3) \in V^*_{reg}\}$$

Example, Theorem, and Problem

$$F_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ F_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

example based on works by Chien and Nakazato (2010), boundary generating surface:

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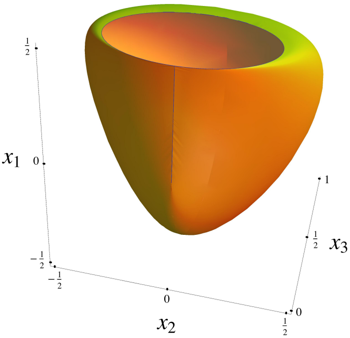

depicted surface:
intersection of $V^*(\mathbb{R})$ with the boundary of W

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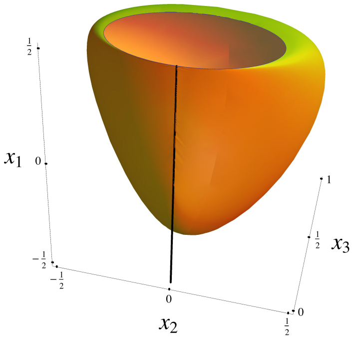

the x_1-axis lies in $V^*(\mathbb{R})$

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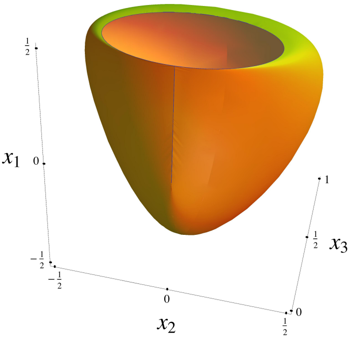

Sinn (2015), Plaumann et al. (2021)

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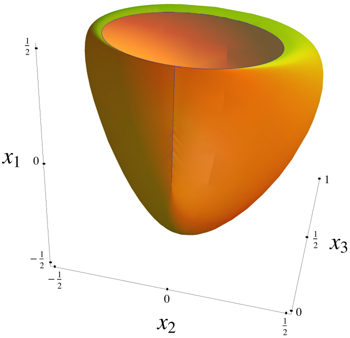

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This result generalizes a Theorem by Kippenhahn (1951) from $l = 2$ matrices to arbitrarily many.

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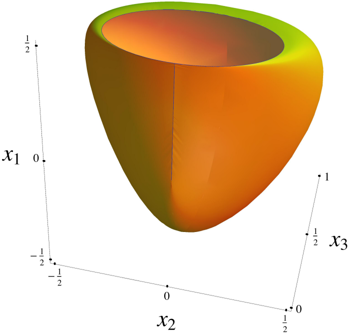

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Problem

Can $V^*(\mathbb{R})$ help detect a lack of lower semicontinuity of the face function of W?

Toric Varieties and Gibbs Varieties

- The inference map restricts to a bijection $\Psi : \mathrm{ri}(\mathcal{D}_1) \to \mathcal{E}$ from the relative interior of $\mathcal{D}_1$ to the Gibbs manifold $\mathcal{E} := \{\exp(A)/\operatorname{Tr}(\exp(A)) : A \in \mathcal{H}_2\}$.

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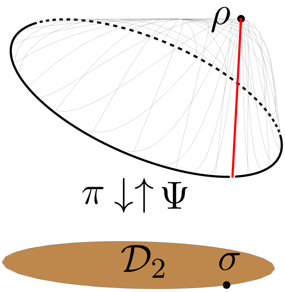
- This fails for the Gibbs variety, the Zariski closure of $\mathcal{E}$, in the quantum domain.



On red: the intersection of the fiber $\pi^{-1}(\sigma)$ with the Euclidean closure of $\mathcal{E}$.

Toric Varieties and Gibbs Varieties

- The inference map restricts to a bijection $\Psi : \mathrm{ri}(\mathcal{D}_1) \to \mathcal{E}$ from the relative interior of $\mathcal{D}_1$ to the Gibbs manifold $\mathcal{E} := \{\exp(A)/\operatorname{Tr}(\exp(A)) : A \in \mathcal{H}_2\}$.
- The fiber $\pi^{-1}(\sigma)$ of every $\sigma \in \mathrm{ri}(\mathcal{D}_1)$ intersects $\mathcal{E}$ in a unique point, the Birch point or Gibbs point (Pavlov et al. 2024).
- If $\mathcal{R}_2$ is generated by diagonal matrices with integer coefficients, then the Zariski closure of $\mathcal{E}$ is known as a toric varietey (Michałek and Sturmfels 2018) and the Birch point is still unique even for boundary points σ of $\mathcal{D}_2$ if the Gibbs manifold is replaced with its toric variety.
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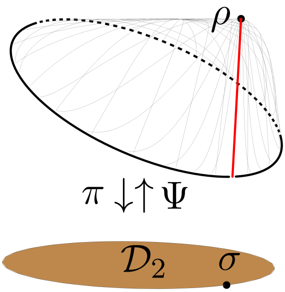

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Problem

Can Gibbs varieties help detect a lack of continuity of the inference map Ψ? Or a lack of openness of the projection π?

Conclusion

- Chen et al. (2015) look for discontinuities of the inference map Ψ in the ground state of a curve of Hamiltonians $H_\epsilon = H_0 + \epsilon H_1$. This connects the discontinuity of Ψ directly to a physical model with energy operator H_ϵ.
- Our approach is model-independent. It can only be a tool to be combined with an actual physical model.

Conclusion

- Chen et al. (2015) look for discontinuities of the inference map Ψ in the ground state of a curve of Hamiltonians $H_\epsilon = H_0 + \epsilon H_1$. This connects the discontinuity of Ψ directly to a physical model with energy operator H_ϵ.
- Our approach is model-independent. It can only be a tool to be combined with an actual physical model.
- It is hoped that algebraic geometry helps understand the lack of openness of partial traces (small local changes lead to global jumps).
- Even heuristic algorithms are appreciated that could find relevant singular points on dual symmetroids and on Gibbs varieties.
- A verification that these singularities are responsible for a lack of openness can be studied using convex geometry and matrix theory.

Thank you for your attention!